BEGINNING ALGEBRA

for College Students

BEGINNING ALGEBRA

for College Students

2nd EDITION

LLOYD L. LOWENSTEIN, Ph.D.
Professor of Mathematics
Kent State University, Kent, Ohio

NEW YORK · JOHN WILEY & SONS, INC.
LONDON · CHAPMAN & HALL, LIMITED

Catalogued
23.1.99

ALLAMA IQBAL LIBRARY
42152

8A=82

J. & K. UNIVERSITY LIB.
Acc. No. 42152
Date 16-2-1966

SECOND EDITION
Second Printing, September, 1954

512
L952 B

Copyright, 1951, by Lloyd Lincoln Lowenstein

Copyright, 1953, by John Wiley & Sons, Inc.

All rights reserved. This book or any part thereof must not be reproduced in any form without the written permission of the publisher.

Library of Congress Catalog Card Number: 53-6135
Printed in the United States of America

To the memory of my father

Albert Lincoln Lowenstein

Preface

During recent years a large number of students with little or no preparation in mathematics beyond eighth grade arithmetic have been entering colleges. Some of them discover that the curricula they wish to follow require mathematics, in particular, college algebra.

There are many good textbooks in college algebra and some more elementary ones with the title *Intermediate Algebra*. Most of these start with a review of elementary algebra. But, as the word implies, reviews are intended merely to remove the fog of remembrance of previous studies. They are not intended to be used as a start in algebra for those who have never been exposed to the subject or for those who were exposed but were not sensitive to it.

This book is a beginning book in algebra. In this sense it duplicates the content of high school algebra textbooks. But it is written with the hope that it will appeal to the more mature mind of the college student.

After studying this book the student should be qualified to enter a course in college algebra. He should be at least as well prepared in algebra as the student who has successfully completed from one and one-half to two years of high school algebra. He should be ready to study the solution of quadratic equations by use of the quadratic formula. He will already have studied the solution of quadratic and even higher-degree equations by factoring.

I believe that rigor and understanding are each part of sound mathematical training. In this book the development of algebra has been made logical and rigorous to an extent not attempted at the high school level. The postulational approach is used, the postulates being suggested by experiences in arithmetic.

Throughout the book I have tried to encourage the student to think for himself. The problems are definitely graded within each exercise. Frequently the later problems of an exercise involve rules which have not yet been discussed. But, if the student thoroughly understands the work up to that point, he himself will discover the rule. For instance, in Exercises 5.1.5, starting with Problem 43, we use the "laws of exponents." But these laws are not formally discussed until several pages later. If the student knows what a^5 and a^3 mean, he has no difficulty in finding $a^5 \cdot a^3$ without having these laws pointed out to him.

Emphasis is also placed upon algebra as a language. In studying a new language we must learn to pronounce the symbols as well as to translate from our known language to the new language and back again. Much practice is given in these three essentials. If the student can learn to pronounce the sounds corresponding to the symbols $(a + b)(a - b)$ so that he knows the difference between these symbols and the symbols $a + b(a - b)$, he will have made considerable progress.

The art of translating from English to algebraic language is introduced early and continued throughout the book. At first no solutions of the resulting equations are called for. But as the student acquires skill in techniques some simple types of equations are solved. For instance, after he has studied subtraction he will be solving equations of the type $3 + x = 7$, not from the point of view of "subtracting 3 from both sides" but from the point of view of using the definition of subtraction. Here $x = 7 - 3$ from this definition. When equations as such are studied he will learn such things as "subtracting equals from equals." He will *not* learn from this textbook such things as "transposing" or "canceling." These words never appear in the book.

It is my opinion that these two "unmentionables" cause more grief to students than any other two words in the mathematical vocabulary. I have never seen the advantage of saying "transpose the 3 to the other side and change its sign" over saying "subtract three from both sides." The first is a mechanical device; the second is an understandable rule. A mechanical device, such as the multiplication tables, is useful to the student who understands the device. But mechanical devices which are perennially misunderstood and misused are to be avoided.

Many students lose sight of the fact that the symbols that we use in algebra stand for numbers. For this reason considerable work with numbers is given in the exercises.

When a student applies his mathematics to problems outside the classroom he will not have someone looking over his shoulder to tell him what data are relevant to the problem at hand. He must judge for himself. Occasionally I have inserted problems containing irrelevant data. See, for instance, Problems 18 and 19 of Exercise 6.4.2.

Also the student should recognize that not every problem that he meets has a solution. The student should learn to recognize when an "answer" makes sense for the problem at hand. This question is discussed in Section 11.10, and thereafter problems for which there are no solutions are inserted.

Years of teaching have convinced me that algebra would frighten fewer people if its relation to arithmetic were grasped. I have written from this point of view, always keeping in mind the age and maturity of the student for whom this book is intended. Although I have made him aware of advanced concepts and the existence of higher levels, the subject matter of this book is algebra and only algebra.

I am indebted to President George A. Bowman of Kent State University for his encouragement. I appreciate the many helpful suggestions given me by the members of the Mathematics Department and by Dean Arden L. Allyn of the College of Business Administration. To my colleague Dr. Arthur E. DuBois of the English Department I am indebted for the removal of many of the rough spots in sentence structure. To Dr. Carl Hammer of the Franklin Institute I am most grateful for his careful reading of the manuscript and many constructive criticisms. Finally I am indebted to the members of my family for their understanding toleration of my neglect of them during the writing.

LLOYD L. LOWENSTEIN

Kent, Ohio
February, 1953

WITH COMPLIMENTS FROM THE ASIA FOUNDATION
NEW

Contents

1 · INTRODUCTION 1

2 · THE NATURAL NUMBERS; ADDITION AND MULTIPLICATION 4

2.1 The Natural Numbers 4
2.2 Postulates for Addition 5
2.3 Symbols of Grouping 8
2.4 Multiplication 10
2.5 Postulates for Multiplication 12

3 · VERBAL PROBLEMS 15

4 · MULTIPLICATION AND ADDITION OF NATURAL NUMBERS 20

4.1 The Distributive Law 20
4.2 Omitting Parentheses 22

5 · POWERS AND EXPONENTS 25

5.1 The Product of n a's 25
5.2 Laws of Exponents 28
5.3 Like Terms 33

6 · SUBTRACTION OF NATURAL NUMBERS 35

6.1 The Inverse Operations 35
6.2 Subtraction 36
6.3 "Greater Than" and "Less Than" 37
6.4 Equivalent Equations 40

7 · ZERO AND THE NATURAL NUMBERS 43

8 · THE INTEGERS; ADDITION AND SUBTRACTION 52

8.1 The Positive and Negative Integers 52
8.2 The Set of Integers 54
8.3 Addition and Subtraction of Integers 56
8.4 Addition of Integers on the Number Scale 57
8.5 Absolute Value; Addition of Integers 59
8.6 Subtraction of Integers 63
8.7 A Rule for Subtraction 65
8.8 Removing Parentheses in Addition and Subtraction Problems 68
8.9 "Greater Than" and "Less Than" for Integers 71

9 · THE INTEGERS; MULTIPLICATION 73

10 · THE INTEGERS; DIVISION 79
- 10.1 Division Defined 79
- 10.2 Order of Operations 82
- 10.3 The Role of Zero in Division 86

11 · FRACTIONS 88
- 11.1 Fish, Farms, and Fractions 88
- 11.2 Equal Fractions 91
- 11.3 Prime Numbers 96
- 11.4 Highest Common Factor 99
- 11.5 Least Common Multiple 100
- 11.6 Least Common Denominator 102
- 11.7 Multiplication of Fractions 103
- 11.8 Fractions and Division 105
- 11.9 Division of Fractions 107
- 11.10 Problems and Their Solutions 111

12 · FRACTIONS; RATIONAL NUMBERS 115
- 12.1 Addition of Fractions 115
- 12.2 Complex Fractions 119
- 12.3 Signed Fractions 123
- 12.4 The Rational Number System 125
- 12.5 Multiplication and Division of Rational Numbers 126
- 12.6 Addition and Subtraction of Rational Numbers 128
- 12.7 "Greater Than" and "Less Than" for Rational Numbers . 131
- 12.8 Some More Laws of Exponents 132

13 · SPECIAL PRODUCTS; POLYNOMIALS 136
- 13.1 A Brief Summary 136
- 13.2 Special Products 137
- 13.3 Constants and Variables 140
- 13.4 Rational Algebraic Expressions; Polynomials 141

14 · FACTORING 144
- 14.1 What Is Factoring? 144
- 14.2 Quadratic Binomials and Trinomials 145
- 14.3 HCF and LCM of Polynomials 151
- 14.4 The Sum and Difference of Two Cubes 158
- 14.5 Factoring by Grouping Terms 160
- 14.6 Factoring of Certain Fourth-Degree Polynomials 161

15 · FUNCTIONS AND GRAPHS 163
- 15.1 Functions 163
- 15.2 Functional Notation 167
- 15.3 Rectangular Coordinates 169
- 15.4 Graphs of Functions 171
- 15.5 Linear Functions 173

16 · OPERATIONS WITH POLYNOMIALS 175
- 16.1 Addition and Subtraction of Polynomials 175
- 16.2 Multiplication of Polynomials 177
- 16.3 Division of Polynomials 180
- 16.4 Decimal Polynomials and Decimal Notation 184

17 · INTEGERS AS EXPONENTS 189
- 17.1 Negative Integers and Zero as Exponents 189
- 17.2 Decimal Fractions 192
- 17.3 Repeating or Periodic Decimals 194
- 17.4 Scientific Notation 196

18 · EQUATIONS AND THEIR SOLUTIONS 199
- 18.1 Identical and Conditional Equations 199
- 18.2 Equivalent Equations 202
- 18.3 Paths to Solutions 210
- 18.4 Equations and Graphs 218
- 18.5 Simultaneous Equations 221

19 · THE REAL NUMBER SYSTEM 227
- 19.1 Irrational Numbers 227
- 19.2 Radicals 230
- 19.3 Tables 232
- 19.4 The Real Number System 234
- 19.5 Rational Exponents and Radicals 235
- 19.6 Multiplication and Division of Radicals 239
- 19.7 Rationalizing Denominators 240

20 · THE COMPLEX NUMBER SYSTEM 243
- 20.1 Pure Imaginary Numbers 243
- 20.2 Complex Numbers 244
- 20.3 Division of Complex Numbers 247

APPENDIX 249

ANSWERS 255

INDEX 273

1

Introduction

In this scientific age most of you will agree that some people should study mathematics. After all, you know that engineers and physicists are needed to build our bridges and roads and to develop improvements in television sets and atomic power. And you have a more or less vague idea that engineers and physicists use mathematics. But a large number of you arrive in college believing that since you are not going to be engineers or physicists you are not included among those people who should study mathematics. You may even be convinced that you could not learn mathematics even if you wished to.

You agree that you should know *some* mathematics, such as arithmetic, so that you will know enough to count your change in the store, or to figure out how much six apples cost if each apple costs 5 cents, or to calculate your income tax. But nothing much beyond that seems necessary, so why rack your brains?

One of the reasons you should study mathematics is just that it will "rack your brains." It will make you think, and what we need in this world more than anything else is people who know how to think.

Another reason for studying mathematics, closely allied to the first, is that we should understand the world about us. Every newspaper has some item in it which can be understood only if we know some mathematics because it uses the language of mathematics. Even the sports pages contain the phrase "the law of averages." More and more it is being realized that man has been most successful in expressing his ideas in mathematical terms, and more and more people are expressing or trying to express their ideas in these terms, even in fields once considered far removed from mathematics.*

---

* See, e.g., Schillinger, Joseph: *Mathematical Basis of the Arts*, Philosophical Library.

We are going to attempt to convince you that you can learn something about this subject. We shall not tell you that it is easy. Nothing which requires the use of the brain is easy. Brain work is the hardest work that you can attempt. But, if you have the raw material, a usable brain, we believe that we can, with your cooperation, start you on the road.

You will have to cooperate by being willing to work. Besides your brain the tools are very simple: a supply of well-sharpened pencils, a pile of scratch paper, and a wastepaper basket. You learn mathematics only by doing it. You cannot learn mathematics by watching the teacher or others solve problems at the blackboard any more than you can learn to play the violin by watching and hearing a concert artist perform.

We can look upon algebra in various ways. It has been called a language—the universal language of the scientist and the principal tool of mathematics above arithmetic; it has been considered from the point of view of logical structure; it has been considered a collection of techniques of calculation; it has been referred to as a collection of puzzles. Algebra is all these.

A complaint which is frequently heard whenever teachers get together "to let down their hair" is that most students do not know how to read. During World War II the comic books were by far the best sellers in army camps. And this was among "the cream of the crop," after more than 240,000 men had been turned down by the army for deficient schooling. We believe that studying mathematics will teach you to read more accurately and critically. When students are asked what they find most difficult in mathematics the answer most frequently given is "word problems." Now word problems are simply problems given to you in the form in which problems arise out in the cold cruel world. Seldom does a problem arise, outside the classroom, in the form of a neat mathematical statement. Learning to translate from spoken and written language to the language of mathematics and back again will be one of our principal objectives. Doing this will help you to read and write and speak our English language with better understanding.

After we have our problems translated into the symbols and language of mathematics we then have the job of solving the problems, using those symbols and that language. This is where the techniques of algebra come in handy.

In order to establish and understand these techniques we must understand and appreciate the logical structure of algebra. This sounds very deep and difficult. But if you will think of algebra as a game played according to fixed rules (called postulates), as any good game must be played, and with certain equipment (called numbers), you will begin to understand what we mean by the logic of algebra. We establish the rules of a game, such as baseball, and play the game according to these rules and with the equipment we agree to use. We do not decide in the middle of a game to change the rules or the equipment even if it would be to our advantage to do so. We take the consequences of our agreements. When we agree to accept a certain rule in mathematics we must agree to accept the consequences of that rule. At times we may bring in additional or new equipment, but that will be done only to make it a better game and it will be done only by mutual agreement among all the players.*

The rules of algebra are actually quite simple. The principal ones are extensions of the rules that you learned (we hope) in arithmetic. In fact we can say that the chief difference between arithmetic and algebra is that in algebra we introduce new equipment which makes the game more interesting and more useful but without discarding the equipment we used in arithmetic. The chief new equipment that we introduce is general symbols (letters) which stand for numbers.

---

* It is possible to play two or more different games with the same equipment but different rules. Bridge and poker are quite different but are played with the same equipment. You might try inventing a game played with the equipment of baseball but which is not baseball. Similarly it is possible to play different "games" with numbers. In ordinary arithmetic with which you are familiar we have $7 + 6 = 13$ and $10 + 5 = 15$. In another kind of arithmetic, called modular arithmetic, we have $7 + 6 = 1$ and $10 + 5 = 3$. This arithmetic might also be called "clock arithmetic." How much is $8 + 8$ in this kind of arithmetic? [1]

Note: Answers to questions referred to by superior numbers in the text are given in the Appendix. However, it will be much more fun and much more instructive if you first try to answer them yourself before looking up the answers.

2

The Natural Numbers; Addition and Multiplication

2.1 The Natural Numbers

In arithmetic our equipment consisted of the so-called natural numbers: 1, 2, 3, 4, 5, $\cdots$ (the three dots are to be read "and so on"). We learned to add, to subtract, to multiply, and to divide these numbers. Addition, subtraction, multiplication, and division are called the arithmetical *operations.* (They have nothing to do with scalpels or anesthetics.) In algebra we shall use these same numbers and operations. But we shall also learn new kinds of numbers and new kinds of operations. You are similar to a person who knows how to play old-fashioned whist * learning to play contract bridge. Fundamentally the games are the same, but many people think contract bridge is much more interesting.

Since you may be out of practice in arithmetic, we are going to start you off by giving you some review problems.

Exercise 2.1.1

Add the numbers in Problems 1 through 20:

1. $\begin{array}{r} 8 \\ \underline{7} \end{array}$	**2.** $\begin{array}{r} 6 \\ \underline{9} \end{array}$	**3.** $\begin{array}{r} 18 \\ \underline{11} \end{array}$	**4.** $\begin{array}{r} 19 \\ \underline{27} \end{array}$
5. $\begin{array}{r} 49 \\ \underline{42} \end{array}$	**6.** $\begin{array}{r} 67 \\ \underline{54} \end{array}$	**7.** $\begin{array}{r} 83 \\ \underline{99} \end{array}$	**8.** $\begin{array}{r} 127 \\ \underline{64} \end{array}$
9. $\begin{array}{r} 255 \\ \underline{138} \end{array}$	**10.** $\begin{array}{r} 751 \\ \underline{269} \end{array}$	**11.** $\begin{array}{r} 699 \\ \underline{742} \end{array}$	**12.** $\begin{array}{r} 809 \\ \underline{463} \end{array}$

---

* For a concise description of whist see *Webster's Unabridged Dictionary.* Most of you are too young to have played this game.

13. $\begin{array}{r}32\\156\\\underline{11}\end{array}$ **14.** $\begin{array}{r}59\\63\\\underline{77}\end{array}$ **15.** $\begin{array}{r}23\\128\\\underline{75}\end{array}$ **16.** $\begin{array}{r}333\\421\\\underline{267}\end{array}$

17. $\begin{array}{r}485\\977\\\underline{862}\end{array}$ **18.** $\begin{array}{r}728\\1463\\\underline{3268}\end{array}$ **19.** $\begin{array}{r}5288\\1492\\\underline{1776}\end{array}$ **20.** $\begin{array}{r}4832\\5764\\9275\\\underline{1234}\end{array}$

Multiply the numbers in Problems 21 through 30:

21. $\begin{array}{r}8\\\underline{7}\end{array}$ **22.** $\begin{array}{r}6\\\underline{9}\end{array}$ **23.** $\begin{array}{r}18\\\underline{7}\end{array}$ **24.** $\begin{array}{r}39\\\underline{5}\end{array}$

25. $\begin{array}{r}7\\\underline{23}\end{array}$ **26.** $\begin{array}{r}23\\\underline{7}\end{array}$ **27.** $\begin{array}{r}233\\\underline{121}\end{array}$ **28.** $\begin{array}{r}199\\\underline{222}\end{array}$

29. $\begin{array}{r}1221\\\underline{11}\end{array}$ **30.** $\begin{array}{r}3574\\\underline{123}\end{array}$

2.2 Postulates for Addition

Now in doing the first twelve of these problems you probably did not raise the question, "Shall I get an answer when I add these two natural numbers and will the answer be a natural number?" The questions may even seem silly. But we are going to put down *all* the postulates (rules) that we are going to use (just as in the rule book for baseball it is stated that the game is played with a ball). We therefore state our

> 2.2.1 Postulate: Given any pair of natural numbers a and b, in that order, there exists a unique (one and only one) natural number called the sum of a and b; this sum is indicated by $a + b$.

This is called the *postulate of closure for addition of natural numbers*.

This postulate actually says more than it appears to say on first reading. The word "closure" is related to the word "closed." Indeed, another way of stating the postulate is "The set of natural numbers is closed under the operation of addition." We may think of all natural numbers placed within a *closed* fence. The postulate says that if we pick out any two of the numbers within this en*closure* and add them we shall find their sum is also within this enclosure.

This is not true of just any set of numbers. For instance suppose that we write down any two numbers from 1 through 10 (the numbers may be the same). The sum of the two numbers we have written down may or may not be a number from 1 through 10. If the numbers written were 2 and 4 or 4 and 4, the sum would be one of the numbers from 1 through 10. But, if we wrote 5 and 6 or 7 and 7, the sum would not be a number from 1 through 10. We say that the set of natural numbers 1 through 10 is *not* closed under the operation of addition.

Is the set of natural numbers closed under the operation of division? [2]

Consider the set of even natural numbers: the set of numbers 2, 4, 6, 8, 10, $\cdots$. Is it closed under the operation of addition? [3]

Consider the set of odd natural numbers: the set of numbers 1, 3, 5, 7, 9, $\cdots$. Is it closed under the operation of addition? [4]

Also in doing the first twelve problems some of you may have done them by "adding down" and others may have done them by "adding up." Some may have said, for Problem 1, 8 plus 7 is 15, and others may have said 7 plus 8 is 15. Again this seems very elementary. But it illustrates our

2.2.2 POSTULATE: If a and b are natural numbers, then $a + b = b + a$.

This is called the *commutative law for addition of natural numbers.* We say that the operation of addition is commutative; you get the same result from either order of addition.

This may also seem to be a trivial statement. Of course, you get the same answer regardless of the order in which you add two numbers (if you do not make a mistake!). It seems less trivial when we stop to realize that many operations are *not* commutative. When you do something in one order you may not get the same result as when you do it in the reverse order. A chemist will tell you that you get a quite different result when you add concentrated sulphuric acid to water from the result you get when you add water to concentrated sulphuric acid. Is the operation of division (in arithmetic) commutative? [5]

We stated earlier that the chief new things that we introduce in advancing from arithmetic to algebra are general symbols (letters) which stand for numbers. You notice that we have already started to do this. We know (or can verify if we know how to add) that

$$2 + 3 = 3 + 2$$

and

$$7 + 5 = 5 + 7$$

and

$$285 + 783 = 783 + 285$$

and

$$1{,}283{,}721 + 5{,}763{,}157 = 5{,}763{,}157 + 1{,}283{,}721$$

and millions of other such statements. How much neater and more elegant to say this about all natural numbers in one simple statement like Postulate 2.2.2!

In the statement of Postulate 2.2.2 the sign = was used. You all know this is read "equals." Many people misuse this sign, and we want to be sure that *you* do not misuse it. We shall use this sign in one way only. If two symbols are connected by the = sign, we shall mean that if in the problem or statement under consideration the symbol to the left of the = sign appears we may, if we want to, replace it by the symbol to the right of the = sign; and if the symbol to the right of the = sign appears we may, if we want to, replace it by the symbol to the left of the = sign. Thus, if $a = b$ and we have the statement "a is an even number," we may also say "b is an even number." This sign (=) is to be used in no other way.*

As we proceed to write down our postulates you may wonder how we decide upon what rules or postulates we are going to use. One way would be simply to make them up as though we were inventing a new game. This is actually done in some advanced branches of mathematics.

However, most of our postulates will be suggested by our experiences. It has been our experience that whenever we add two natural numbers the result is independent of the order in which we add them. Hence we formulate Postulate 2.2.2. We cannot prove this postulate, nor any other postulate, but we certainly should be surprised if we found an example for which it did not hold. Our postulates will agree with our experiences in working with the natural numbers in arithmetic.

---

* The sign = for "equals" was introduced by Robert Recorde in his algebra book *The Whetstone of Witte* (1557). He explained his invention of the equals sign by writing "a pair of parallel lines of one length, thus =, because no two things can be more equal." Also in this book the + and − signs for addition and subtraction appeared for the first time in a book written in the English language.

Another experience which we had in arithmetic is illustrated by Problems 13 through 19 of Exercise 2.1.1. For instance, in adding the first column in Problem 13 you may have said to yourself "2 and 6 is 8 and 1 is 9" or you may have said "6 and 1 is 7 and 2 is 9." This common experience in adding three natural numbers suggests

2.2.3 POSTULATE: If a, b, and c are natural numbers, then $(a + b) + c = a + (b + c)$.

This is called the *associative law for addition of natural numbers.*

2.3 Symbols of Grouping

We are going to take time out here to make sure that you know how to read the equality contained in Postulate 2.2.3. Your objective at all times in reading a mathematical sentence (an equality is a mathematical sentence) should be so to read it that if you were reading it to a friend over the telephone the friend would be able to reproduce it on a piece of paper. The equality $(a + b) + c = a + (b + c)$ is read "the quantity a plus b, plus c equals a plus the quantity b plus c."

2.3.1 *Parentheses indicate that the symbols within them are to be considered together as one quantity.*

In reading symbols enclosed in parentheses we start with the words "the quantity" or simply say "parentheses" and then "close parentheses" when we should. When we start with "the quantity" and there is any doubt where this quantity ends we may say "end quantity" at the proper place. Sometimes we need to have parentheses within parentheses; in order not to confuse ourselves we use different symbols of grouping such as [ ], called brackets, and { }, called braces. More rarely we use a vinculum, which is a line drawn over all the symbols which are to be considered as one quantity, as $\overline{a + b}$.*

---

* Parentheses, brackets, and braces as symbols of grouping were rather late additions to our symbols. They came into use in the early eighteenth century. They became popular through their use by the great Swiss mathematician

Exercise 2.3.2

Read each of the following. (Be sure you would understand if you heard it over the telephone.)

1. $(a + b) + c = c + (a + b)$.
2. $(a + c) + b = (b + a) + c$.
3. $(b + a) + c = (c + b) + a$.
4. $\overline{a + b} + c = a + \overline{b + c}$.
5. $(a + b) + (c + d) = (c + d) + (a + b)$.
6. $(a + b) + (c + d) = a + [b + (c + d)]$.
7. $[(a + b) + c] + d = a + [b + (c + d)]$.
8. $a + [(b + c) + d] = (a + c) + (b + d)$.
9. $\{a + [(b + c) + d]\} + e = [a + (b + c)] + (d + e)$.
10. $\{a+[b+(c+\overline{d+e})]\}+f=[a+(b+c)]+[(d+e)+f]$.

Write each of the following in mathematical symbols.

11. The quantity a plus b, plus c equals c plus the quantity a plus b.
12. The quantity a plus the quantity b plus c, plus d equals a plus the quantity b plus the quantity c plus d.
13. Bracket, the quantity a plus b, plus c, close bracket, plus d equals a plus the quantity b plus the quantity c plus d.
14. a plus bracket, the quantity b plus c, plus d, close bracket, equals the quantity a plus b plus the quantity c plus d.

Problems 1 through 10 of Exercise 2.3.2 may be considered consequences of our first three postulates. For instance in Problem 5, by the postulate of closure for addition of natural numbers, $a + b$ and $c + d$ are natural numbers, and hence by the commutative law for addition of natural numbers we may add these in either order. In general,

> *in any addition problem, the sum of any number of natural numbers is a natural number; we may add them in any order we please and we may group them in any way we please.*

Leonhard Euler. Previously the vinculum was the most used symbol of grouping. Vincula present a problem in typesetting since they must be placed by hand. When all type was hand set this presented no special problem. But with the coming of machine processes in printing it became essential that symbols follow each other in a line as much as possible.

Because of this, in a sum such as $\{a + [(b + c) + d]\} + e$, we may, if we wish, omit all symbols of grouping and obtain $a + b + c + d + e$; we may likewise in a sum such as $a + b + c + d + e$ insert symbols of grouping in any way that makes sense, such as $[(a + b) + (c + d)] + e$; and we may rearrange the order in any way we please such as $c + a + b + e + d$.

A grouping which does *not* make sense is $(a + [b + c) + d] + e$. This does not make sense because the parentheses tell us that $a + b + c$ is one quantity while the brackets say that $b + c + d$ is one quantity. Symbols of grouping must not overlap.

Exercise 2.3.2 (continued)

15 through 21. Show, as was done above for Problem 5, that the statements in Problems 1, 2, 3, 4, 6, 7, and 8 are consequences of our first three postulates.

22. Do Problem 20 of Exercise 2.1.1 by adding the first two numbers and then the last two and then adding these sums.

Calculate the following, that is, write each as a single natural number. First make your calculation by following the order indicated by the symbols of grouping, and then remove the symbols of grouping and add in order from left to right.

23. $3 + (4 + 5)$.
24. $3 + [4 + (5 + 6)]$.
25. $[2 + (3 + 4)] + 5$.
26. $(2 + 4) + (3 + 5)$.
27. $[(5 + 2) + 3] + 4$.
28. $2 + \{3 + [4 + (5 + 6)]\}$.
29. $\{[2 + (3 + 4)] + 5\} + 6$.
30. $[(2 + 3) + (4 + 5)] + 6$.

31. $[(4 + 2) + 3] + (6 + 5)$.

32. $(3 + \overline{4 + 2}) + (\overline{5 + 6} + 7)$.

2.4 Multiplication

Now that we know something about addition of natural numbers, let us consider multiplication. Of course, you know your multiplication tables! But we want to build up our game so that anyone can learn to play it from our rules.

First we had better state what we mean by multiplication of natural numbers. This means that we must make up a definition. (If you were writing a rule book for baseball, how would you tell what a strike is? This would be a definition of strike.)

When a number a is added to itself we get $a + a$ or $2a$. When we have $a + a + a$ we get $3a$; $a + a + a + a$ gives us $4a$. Instead of reading these as $2a$, $3a$, $4a$ we could say 2 times a, 3 times a, 4 times a or the *product* of 2 and a, etc. This suggests the

2.4.1 DEFINITION: If a and b are natural numbers, then the *product* of a and b means $b + b + \cdots + b$ where there are a terms in the sum. This product is written $a \times b$ or $a \cdot b$ or ab. To multiply a and b means to find the product ab; a and b are said to be the *factors* of the product ab; ab may also be read "a times b."

The first symbol for the product of a and b, $a \times b$, is probably the one you used in arithmetic. In arithmetic there was no doubt of what you meant when you wrote 5×9. But, since we are now using letters as well as numerals to stand for numbers and one of the most popular letters in algebra is x, this may be misunderstood. It would be easy to confuse the multiplication sign $\times$ with the letter x. From now on we shall never use the cross symbol for multiplication.

EXAMPLES 2.4.2

Let us write down a few numbers which are to be multiplied.

1. a times b may be written $a \cdot b$ or it may be written simply as ab.
2. 3 times p may be written $3 \cdot p$ or it may be written simply as $3p$.
3. 3 times 5 *must* be written as $3 \cdot 5$. Otherwise it would look like 35 which is the number thirty-five and not 3 times 5 or 15.
4. a times the quantity b plus c may be written $a \cdot (b + c)$ or it may be written simply as $a(b + c)$.
5. 5 times the quantity 3 plus p may be written $5 \cdot (3 + p)$ or it may be written simply as $5(3 + p)$.
6. x times bracket y plus 3 times the quantity p plus 4, close bracket, may be written as $x \cdot [y + 3 \cdot (p + 4)]$ or $x[y + 3(p + 4)]$.

Exercise 2.4.3

Read each of the following: (Remember that friend on the other end of the line!)

1. $11 \cdot 35$.
2. $75a$.
3. $5(ab)$.
4. $5 \cdot (7q)$.
5. $5(7q)$.
6. $x(yz)$.
7. $a(xb)$.
8. $8 \cdot (r + s)$.
9. $5(4 + t)$.
10. $25 \cdot (10 + 1)$.
11. $a + b(c + d)$.
12. $(a + b)(c + d)$.
13. $m[(n + 1) + x]$.
14. $m[n + (1 + x)]$.
15. $[p + (2q + r)] \cdot [(p + 2q) + r]$.
16. $\{a + [2 + (3 + b)]\} \cdot [(2 + a) + b]$.

Write each of the following in mathematical symbols:

17. Twenty-five times seven.
18. Nineteen times q (two ways).
19. Eleven times the quantity three s.
20. Eleven times the quantity three plus s.
21. The quantity r plus s, times the quantity x plus y.
22. Bracket a plus the quantity b plus c, close bracket, times bracket, the quantity a plus b, plus c, close bracket.

2.5 Postulates for Multiplication

By considering Problems 21 through 30 of Exercise 2.1.1 we are led to

2.5.1 Postulate: If a and b are natural numbers, in that order, there exists a unique natural number called the product of a and b.

This is called the *postulate of closure for multiplication of natural numbers*.

Also by noting that $8 \cdot 7 = 7 \cdot 8$ and similar equalities it is suggested that we accept the

2.5.2 Postulate: If a and b are natural numbers, then $ab = ba$.

What might be the name of this rule?[6]

Since $3\cdot(4\cdot5) = 3\cdot20 = 60$ and also $(3\cdot4)\cdot5 = 12\cdot5 = 60$, we decide to formulate

2.5.3 POSTULATE: If a, b, c are natural numbers, then $a(bc) = (ab)c$.*

What might be the name of this postulate?[7]

EXERCISE 2.5.4

Show that each of the following follows from, or is a consequence of, our postulates:

1. $pq = qp$.
2. $(3 + 4)\cdot5 = 5\cdot(3 + 4)$.
3. If x, y, and z are natural numbers, then $x(yz)$ is a natural number.
4. $7(xy) = (7x)y$.
5. $3 + (4\cdot5) = (4\cdot5) + 3$.
6. $4[(5 + 6)(6 + 7)] = [4(5 + 6)]\cdot(6 + 7)$.
7. $a[(b + c)(d + e)] = [a(b + c)]\cdot(d + e)$.
8. $(5\cdot7) + 2 = 2 + (7\cdot5)$.
9. $(ab)(cd) = (cd)(ab)$.
10. $(ab)(cd) = a[b(cd)]$.
11. $c[(ab)d] = (ad)(bc)$.
12. $\{a[(bc)d]\}e = [a(bc)]\cdot(de)$.

Calculate each of the following:

13. $(3 + 4)\cdot5$.
14. $3 + (4\cdot5)$.
15. $(3 + 4)(4 + 5)$.
16. $3 + [(4\cdot5) + 6]$.
17. $[3 + (4\cdot5)] + 6$.
18. $3 + [4\cdot(5 + 6)]$.
19. $[2 + (3\cdot4)]\cdot(5 + 3)$.
20. $2[(2 + 3) + (4 + 5)]$.
21. $2\{[(2 + 3) + 4] + 5\}$.
22. $2\{[2 + (3 + 4)] + 5\}$.
23. $2\cdot\{2 + [(3 + 4) + 5]\}$.
24. $2\cdot\{(4 + 5) + (2 + 3)\}$.

Problems 9 through 12 of this exercise are consequences of Postulates 2.5.1, 2.5.2, and 2.5.3. In general,

in any multiplication problem the product of any number of natural numbers is a natural number; we may multiply them in any order we please and we may group them in any way we please.

---

* Note: We are in no sense stating that Postulate 2.5.3 follows from our example. This example and others like it suggest that this postulate would be a good rule for our game.

Because of this, in a product such as $a\cdot\{[(b\cdot c)\cdot d]\cdot e\}$ we may, if we wish, omit all parentheses and obtain $a\cdot b\cdot c\cdot d\cdot e$ or simply $abcde$; we may likewise in a product such as $abcde$ insert parentheses in any way that makes sense, as $[(ab)(cd)]e$, and we may rearrange the order in any way we please, as $cabed$.

Calculate each of the following. First make your calculation by following the order indicated by symbols of grouping, and then remove all symbols of grouping and multiply in order from left to right.

25. $3(4\cdot 5)$.
26. $2[3(4\cdot 5)]$.
27. $[2(3\cdot 4)]5$.
28. $(2\cdot 3)(4\cdot 5)$.
29. $[(5\cdot 2)3]4$.
30. $2\{3[4\cdot(5\cdot 6)]\}$.
31. $\{[2(3\cdot 4)]5\}6$.
32. $[(2\cdot 3)(4\cdot 5)]6$.
33. $[(4\cdot 2)3](6\cdot 5)$.

34. $(3\cdot\overline{4\cdot 2})(\overline{5\cdot 6}\cdot 7)$.

Calculate each of the following:

35. $2(3+4)[5(2+4)]$.
36. $5[1+\overline{2(3+4)}][3(2+3)]$.
37. $[\overline{2(3+4)}+\overline{3(2+3)}]+3(1+5)[2+(1+3)]$.
38. $\{[(1+2)+3][2+\overline{3(1+4)}]\}+5\{1+[\overline{2(3+4)}+5]+2\}$.

3

Verbal Problems

3.1 Translation from Words to Algebra

We have said that the chief new things that we introduce in algebra are letters which stand for numbers. Of course, if we want to talk about a specific number, such as four, we write 4. But, as in our postulates, we frequently want to make statements which are to hold for all numbers, and in this case we use a letter to represent *any* number. Another instance in which we use a letter to stand for a number is that in which the number is unknown.

If a murder is committed and the murderer is unknown, a warrant may be issued for his arrest. The warrant may be issued for John Doe. Until his true identity is known he is known by this name.

Many problems arise in the sciences and elsewhere which are of the "John Doe" type. We are looking for an unknown. It has been found that algebra is often a big help in finding the culprit in those cases in which the unknown is a number. Until we learn his true identity we call him by a name x or d or any other handy letter.*

Before we start the search, using algebra to help us, we must first be able to state the problem in the language of algebra. In other words we must learn how to *translate* the problem from English to the language of algebra. We shall illustrate with some examples. In these examples we should remember that a letter represents a *number*. For instance, if we say "let x represent John's age," we mean that x represents the number of years in John's age; or, if we say "let s represent the length of a rectangle," then s represents the number of units of length (feet, yards, etc.).

---

* The use of x, y, z to represent unknowns is due to René Descartes. They appeared in his book *La géométrie* (1637).

EXAMPLES 3.1.1

1. A certain number is six more than eight. Suppose that we call this unknown number by the name n. We are told that n is six more than eight. In algebraic language we are saying $n = 8 + 6$.

2. The sum of a certain number and twice that number is 18. Let us call this unknown number p. We are told that the sum of p and twice p is 18. We obtain the algebraic sentence $p + 2p = 18$.

3. If we add six to a certain number, we obtain four times that number. Let us call the number x. We are told that, if we add six to x, we obtain $4x$. We have then $x + 6 = 4x$.

4. If we take four times three more than a certain number, we obtain 20. Let us call this number y. We are told that, if we add three to y and multiply the result by four, we obtain 20. Our algebraic sentence is $4(y + 3) = 20$.

5. A father is 24 years older than his son, and 10 years from now he will be exactly twice as old as his son. Let us call x the son's present age. Then $x + 24$ is the father's present age, $x + 10$ will be the son's age in 10 years, and $(x + 24) + 10$ will be the father's age in 10 years. We are told that the father's age in 10 years will be twice the son's age then. We obtain $(x + 24) + 10 = 2(x + 10)$.

6. Forty-three dollars are divided among Tom, Dick, and Harry. Tom receives twice as much as Dick, and Harry receives $3 more than Tom. We let d be the number of dollars Dick receives. Then $2d$ is the number of dollars Tom receives, and then $2d + 3$ is the number of dollars Harry receives. Then, since the total number of dollars divided is 43, we have $d + 2d + (2d + 3) = 43$.

In these examples we have illustrated the process of translating from English to algebraic language. The result is an equation. We call this "setting up the equation for the problem."

However, in none of these examples have we "solved the equation," as finding the value of the unknown is called. Setting up the equation and solving the equation are two separate processes. The process of setting up the equation requires careful reading and thought. The process of solving the equation requires knowledge of and practice in algebraic techniques. We have not yet discussed enough algebraic techniques to solve any but the simplest equations, such as that we set up in Example 1.

No one method of setting up equations can be given. Your ingenuity is needed. We can give general advice only.

1. Read the problem thoroughly to get the general idea.
2. Read it again so as to understand it thoroughly.
3. Decide what is "unknown" and give it a name, some letter such as x, d, p, etc. Write down a statement of what your unknown is and what you are calling it.
4. Write down each of the other quantities involved in the problem in terms of your "unknown." *
5. Write an equality between the quantities which the problem states are equal.

EXERCISE 3.1.2

Translate each of the following into an algebraic sentence (equation):

1. Seven more than a certain number is 15.
2. If twice a certain number is increased by one, we obtain 43.
3. The sum of a number and three times that number is 56.
4. Four times three more than a certain number is 36 more than the number.
5. Joe is twice as old as John, and the sum of their ages is 42.
6. Mike would be 36 years old if he were twice as old as he will be in 3 years.
7. A piece of cord 42 feet long is cut so that one piece is twice as long as the other.
8. We have three numbers of which the first is twice the second and the third is 4 more than the first; their sum is 59.
9. A cord 59 inches long is cut into three pieces of which the first is twice as long as the second and the third is 4 inches longer than the first.
10. Dick has twice as many marbles as Tom, and Sam has four more than Dick. Together they have 59 marbles.

---

* For quite some time each problem will have just one unknown or, if more than one, the others can be expressed readily in terms of one. For instance, in Example 5 we might say that both the son's age and the father's age are unknowns. But the father's age is easily obtained, once we have called the son's age x.

11. A man has $80 in $1, $5, and $10 bills, the same number of each denomination.

By consecutive natural numbers we mean a group of natural numbers taken in the order in which they appear in 1, 2, 3, 4, 5, · · ·. For instance, 4, 5, 6 are three consecutive natural numbers.

12. The sum of three consecutive natural numbers is 18.

13. The product of two consecutive natural numbers is 132.

14. Four consecutive natural numbers are such that the product of the first and last plus the product of the other two is 82.

15. We learn in geometry that the sum of the angles of a triangle is 180 degrees. One angle of a triangle is 30 degrees; the second is twice the third.

16. A father earns twice as much each day as his son. The father works 8 days and the son works 5 days. They receive together payment of $84 for their work. How much does each earn each day?

17. The length of a rectangle is twice its width. Its perimeter is 66 feet. What are its dimensions?

18. The length of a rectangle is 4 feet more than twice its width. Its area is 240 square feet. What are its dimensions?

19. A rectangular field is four times as long as it is wide. If it were 5 yards longer and 2 yards wider, its area would be increased by 140 square yards.

20. The perimeter of a rectangle is three times its length. Its width is 12 feet.

21. The perimeter of a rectangle is five times its width. Its length is 60 feet.

22. The number of square feet in the area of a rectangle is the same as the number of feet in its perimeter. The width of the rectangle is five feet.

23. At the school election a total of 454 votes was cast for the two candidates for president, Albert and Paul. Albert won by 60 votes. How many votes did Paul get?

24. A ball is thrown downward from a balloon with an initial speed of 40 feet per second. Each second that the ball falls its speed increases by 32 feet per second. After a number of seconds its speed is 168 feet per second.

25. A motor boat can travel at a top speed of 10 miles per hour. It crosses a lake at that speed, and just as it starts back something goes wrong with the motor so that it then can go only 3 miles per hour. It takes 6 hours longer for the return trip than it did for the original crossing. How long did it take to make the original crossing?

26. The population of a town is 50,000 and is increasing by 700 each year. Another town has a population of 29,000 and its population is increasing 1000 each year. If this continues, after a number of years the towns will have the same population.

4

Multiplication and Addition of Natural Numbers

We have said that we have not yet discussed enough algebraic techniques to solve any but the simplest equations. Let us now set out to acquire some of these needed techniques.

4.1 The Distributive Law

We shall consider another postulate which is useful in handling expressions such as $5(3 + 4)$. We note that this could be evaluated by saying $5(3 + 4) = 5\cdot 7 = 35$. But we also easily see that $5(3 + 4) = (5\cdot 3) + (5\cdot 4) = 15 + 20 = 35$. This suggests

4.1.1 POSTULATE: If a, b, c are natural numbers then

$$a(b + c) = (ab) + (ac).$$

This is called the *distributive law for natural numbers.* It tells us that the multiplier a is *distributed* over the terms of the sum $b + c$.

The equality given in this postulate is very useful in some cases when read from left to right and in other cases when read from right to left.

EXAMPLES 4.1.2

1. $5(6 + 7) = (5\cdot 6) + (5\cdot 7) = 30 + 35 = 65.$
2. $(4\cdot 7) + (4\cdot 9) = 4(7 + 9) = 4\cdot 16 = 64.$
3. $x(y + z) = \overline{xy} + \overline{xz}.$
4. $\overline{pq} + \overline{pr} = p(q + r).$
5. $$\begin{aligned}(a + b)(c + d) &= [(a + b)c] + [(a + b)d]\\ &= [c(a + b)] + [d(a + b)] \quad \text{(Why?)}\\ &= (\overline{ca} + \overline{cb}) + (\overline{da} + \overline{db})\\ &= \overline{ca} + \overline{cb} + \overline{da} + \overline{db}\\ &= \overline{ac} + \overline{bc} + \overline{ad} + \overline{bd}\end{aligned}$$

Jammu & Kashmir University Library,
Accession No. 47152

or

$$\begin{aligned}(a+b)(c+d) &= (c+d)(a+b)\\ &= [(c+d)a] + [(c+d)b]\\ &= [(ac)+(ad)] + [(bc)+(bd)]\\ &= (ac)+(ad)+(bc)+(bd).\end{aligned}$$

6.
$$\begin{aligned}(xp)+(xq)+(yp)+(yq) &= [(xp)+(xq)] + [(yp)+(yq)]\\ &= [x(p+q)] + [y(p+q)]\\ &= (p+q)(x+y)\end{aligned}$$

or

$$\begin{aligned}(xp)+(xq)+(yp)+(yq) &= (xp)+(yp)+(xq)+(yq)\\ &= [(xp)+(yp)] + [(xq)+(yq)]\\ &= [p(x+y)] + [q(x+y)]\\ &= (x+y)(p+q).\end{aligned}$$

7.
$$\begin{aligned}a(x+y+z) &= a[(x+y)+z]\\ &= [a(x+y)] + (az)\\ &= (ax)+(ay)+(az).\end{aligned}$$

8.
$$\begin{aligned}(as)+(at)+(ar)+(bs)+(bt)+(br)\\ &= [\overline{as} + (\overline{at} + \overline{ar})] + [\overline{bs} + (\overline{bt} + \overline{br})]\\ &= [\overline{as} + \overline{a(t+r)}] + [\overline{bs} + \overline{b(t+r)}]\\ &= \{a[s+(t+r)]\} + \{b[s+(t+r)]\}\\ &= (s+t+r)(a+b).\end{aligned}$$

1, 3, 5, and 7 of the above examples illustrate the use of the distributive law in "multiplying out." 2, 4, 6, and 8 illustrate its use in "taking out the common factor."

5, 6, 7, and 8 may be considered as extensions of the distributive law. In general

$$a(p+q+\cdots+z) = \overline{ap} + \overline{aq} + \cdots + \overline{az}$$

and

$$\begin{aligned}&(a+b+\cdots+m)(p+q+\cdots+z)\\ &\qquad = (a+b+\cdots+m)p + (a+b+\cdots+m)q\\ &\qquad\qquad + \cdots + (a+b+\cdots+m)z.\end{aligned}$$

Exercise 4.1.3

Multiply out:

1. 9(10 + 9).
2. 8(300 + 50 + 7).
3. 19(200 + 60 + 5).
4. (10 + 9)(20 + 5).

5. $(10 + 9)(600 + 70 + 8)$.
6. $(200 + 30 + 5)(500 + 90 + 4)$.

7. $x(y + z + w)$.
8. $5x(y + z + w)$.
9. $5x(2y + 3z + 4w)$.
10. $(x + y)(p + q)$.
11. $(x + y)(p + q + r)$.
12. $(x + y + z)(p + q + r)$.

13. $(x + y + z)(p + q + r + s)$.
14. $(x + y + z + w)(p + q + r + s)$.
15. $[(2a) + (3b)][(4m) + (5n)]$.
16. $[(2a) + (3b) + (4c)][(4m) + (5n)]$.

17. $s[a + \overline{b(c + d)}]$.
18. $s[\overline{(a + b)c} + d]$.
19. $2t[(3s) + \overline{2(w + v)}]$.
20. $(x + y)(x + y)$.
21. $(x + y + z)(x + y + z)$.
22. $[(2a) + b][(2a) + b]$.

23. $[(2a) + (3b)][(2a) + (3b)]$.
24. $(3p) + \{(2q)[(3p) + (2q)]\}$.
25. $[(3p) + (2q)][(3p) + (2q)]$.
26. $[(5m) + (2n)] + \{7s[(5m) + (2n)]\}$.

Take out the common factor:

27. $(7 \cdot 5) + (7 \cdot 9)$.
28. $(7a) + (7b)$.
29. $(7a) + 7$.
30. $(7a) + (7a)$.
31. $(xy) + (xz)$.
32. $(xy) + (xz) + (xw)$.
33. $(2xy) + (2xz) + (2xw)$.
34. $(3xy) + (6xz)$.

35. $(xy) + (xz) + (py) + (pz)$.
36. $(27xy) + (18yz)$.
37. $(3xy) + (6xz) + (5yw) + (10zw)$.
38. $(xxy) + (xyy)$.

4.2 Omitting Parentheses

We have seen that in problems involving addition only and in problems involving multiplication only we may, if we wish, omit parentheses. In certain cases, by making a simple agreement, we also may omit parentheses in problems involving both addition and multiplication. This agreement is:

4.2.1 *In any problem involving both addition and multiplication the multiplications are to precede the additions unless parentheses indicate otherwise.*

EXAMPLES 4.2.2

1. $3 + 4 \cdot 5 + 6 = 3 + 20 + 6 = 29$.

2. $(3 + 4)5 + 6 = 7 \cdot 5 + 6 = 35 + 6 = 41$.

3. $(3 + 4)(5 + 6) = 7 \cdot 11 = 77$.

4. $a + bc + d = a + (bc) + d$.

5. In $(a + b)c + d$ we may not omit the parentheses except by applying the distributive law. Thus

$$(a + b)c + d = (ac) + (bc) + d = ac + bc + d.$$

6. $$\begin{aligned}(1 + 2) + 3[4(5 + 6) + 7 \cdot 8] &= (1 + 2) + 3[(4 \cdot 11) + 7 \cdot 8] \\ &= 3 + 3[44 + 56] \\ &= 3 + 3 \cdot 100 \\ &= 3 + 300 = 303\end{aligned}$$

or

$$\begin{aligned}(1 + 2) + 3[4(5 + 6) + 7 \cdot 8] &= 3 + 3[4 \cdot 5 + 4 \cdot 6 + 7 \cdot 8] \\ &= 3 + 3 \cdot 20 + 3 \cdot 24 + 3 \cdot 56 \\ &= 3 + 60 + 72 + 168 = 303.\end{aligned}$$

EXERCISE 4.2.3

Calculate (some of these can be done in more than one way):

1. $2 + 3 \cdot 4 + 5$.

2. $(2 + 3)4 + 5$.

3. $(2 + 3)(4 + 5)$.

4. $2 + 3(4 + 5)$.

5. $2 + (3 + 4)5$.

6. $2 + 3 \cdot 4 + 5[6 \cdot 7 + 8(9 + 10)]$.

7. $(2 + 3)4 + 5[6 + 7 \cdot 8(9 + 10)]$.

8. Since $2 + 3 \cdot 4 = 2 + 12 = 14$ and $2(3 + 4) = 2 \cdot 7 = 14$, may we conclude that it would be a good idea to make $a + bc = a(b + c)$ a postulate?

Remove all parentheses:

9. $[a(b + c)] + \{[d + (e + f)] + (gf)\}$.

10. $[x + y(z + 2s)] + \{3[2x + 2z(y + s)]\}$.

Although you should try to keep your notation as simple as possible, you must not act as though there were a shortage of parentheses and omit them when their omission may cause misunderstanding. It is better to have more parentheses than necessary than to have fewer than are needed to state what you wish to state. In the first case all that is wasted is some lead from your pencil and a little time. In the second case someone, perhaps you yourself, will not read what you have written as you intended it to be read. For instance, if you want to have the quantity a plus b times the quantity c plus d it is not correct to write $a + b \cdot c + d$.

Remove only unnecessary parentheses (without performing any operations):

11. $[3 + (4 \cdot 5)] + 6$.
12. $[(3 + 4)5] + 6$.
13. $[3 + (4 \cdot 5)] + \{[(6 \cdot 7)(8 + 9) + (2 \cdot 3)] + 4\}$.
14. $[a(b + c)] + \{[d + (e + f)] + (gf)\}$.
15. $\{s[x(y + z)]\} + s\{[(x + y)(y + z) + x] + (yz)\}$.
16 through **26.** Remove all unnecessary parentheses from Problems 16 through 26 of Exercise 4.1.3.

5

Powers and Exponents

5.1 The Product of n a's

When we set out to define multiplication we noted that $a + a = 2a$, $a + a + a = 3a$, $a + a + a + a = 4a$, etc. In other words, the sum of n a's was indicated by na.

Suppose that we now consider the product of n a's. Of course, a times a could be written as aa; a times a times a could be written as aaa. But, if we had the product of fifteen a's, it would be very awkward written in this form. We therefore decide to invent a new notation for this. This notation is a^{15} and is called the fifteenth power of a or a to the fifteenth power.

5.1.1 DEFINITION: If a and n are natural numbers, then a to the nth power means $a \cdot a \cdot a \cdot \cdots \cdot a$ where there are n factors in the product; it is written a^n. To raise a to the nth power means to find a^n; a is called the base, n is called the exponent, and a^n is called the nth power of a. When n is one we write $a^1 = a$.*

We should always remember that when a and n are natural numbers a^n is simply a symbol for a special kind of product, namely the product of n a's, just as $n \cdot a$ is a symbol for a special kind of sum, namely the sum of n a's.

If you did Problem 20 of Exercise 4.1.3 correctly, your answer was $xx + 2xy + yy$. With our new notation this is written $x^2 + 2xy + y^2$. The answers to Problems 21 through 25 of that exercise also may be rewritten with this notation.

A geometrical square has long been a familiar figure. If the sides of a square are x units long, its area is $x \cdot x$ or x^2. Because of this a number raised to the second power has come to be called x squared. We speak of "the square of x," meaning x^2.

---

* Credit for inventing the symbol a^n for the product of n a's is generally given to René Descartes. They first appeared in his book *La géométrie* (1637).

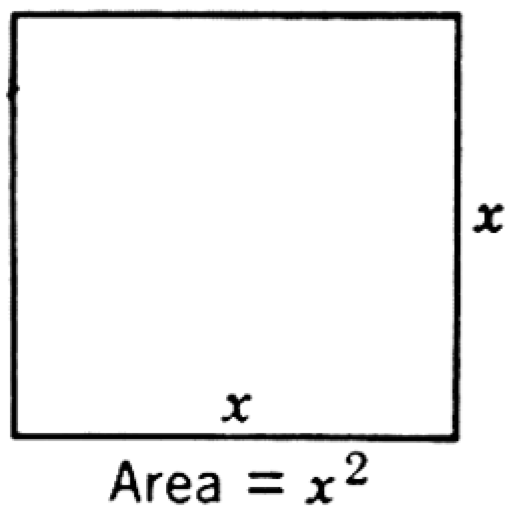


FIG. 5.1.2.

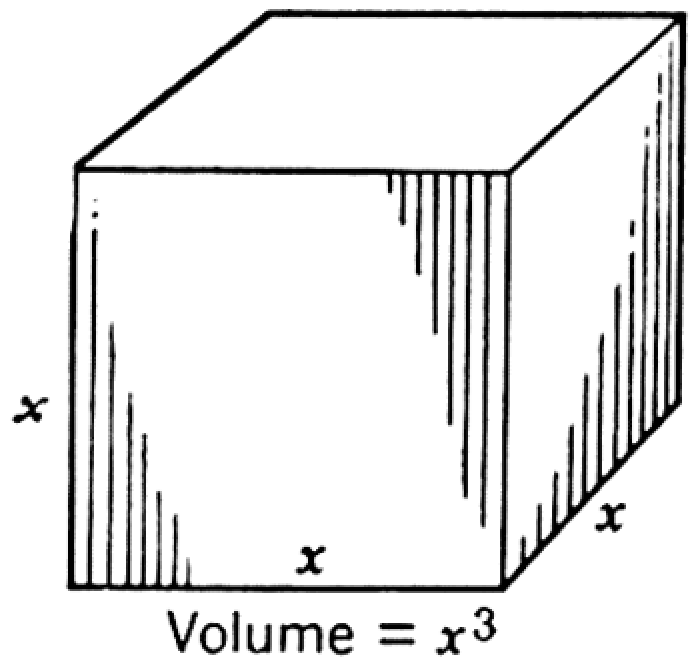


FIG. 5.1.3.

Similarly a cube with edge x has volume x^3. We give this particular power of x the name x cubed. We speak of "the cube of x," meaning x^3.

EXAMPLES 5.1.4

1. $p \cdot p \cdot p = p^3$ (read p cubed or the cube of p).
2. $a \cdot a \cdot b \cdot b \cdot b = a^2 \cdot b^3 = a^2b^3$ (read a squared times b cubed or simply a squared b cubed).
3. $x \cdot x \cdot x \cdot y \cdot y \cdot y \cdot y = x^3y^4$ (read x cubed times y to the fourth power or simply x cubed y fourth).
4. $(a + b)^2$ is read "the quantity a plus b, squared" or "the square of the quantity a plus b."
5. $(p + q + r)^4$ is read "the quantity p plus q plus r to the fourth power."
6. $[x + (y + z)]^2$ is read "bracket, x plus the quantity y plus z, all squared" or "the square of bracket, x plus the quantity y plus z, close bracket."

Note that an exponent refers only to the quantity *immediately* preceding it. In xy^3 only the y is "cubed"; that is, this means $x \cdot y \cdot y \cdot y$; in $a(b + c)^2$ only the $b + c$ is squared; that is, this means $a(b + c)(b + c)$; $(xy)^3$ means $(xy)(xy)(xy)$.

EXERCISE 5.1.5

Read each of the following:

1. x^2y^3.
2. $x^2y^4z^3$.
3. xy^3.
4. $(xy)^3$.
5. $2x^2$.
6. $(a + b)^2(x + y)^3$.

7. $(a + b^2)(x + y^3)$.
8. $(3x + y)^2(4u + v)^3$.
9. $(x + 2y + z)^5(x + y)^4$.
10. $(2x)^2 = 4x^2$.
11. $x^3 \cdot x^4 = x^7$.
12. $(x^2)^3 = x^6$.
13. $(xy)^5 = x^5y^5$.
14. $(a^2b^3)^4 = a^8b^{12}$.

15. $(x + y)^2 = x^2 + 2xy + y^2$.
16. $(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$.
17. $(2a + b)^2 = 4a^2 + 4ab + b^2$.
18. $(a + 2b)^2 = a^2 + 4ab + 4b^2$.
19. $(x^2 + y^2)^2 = x^4 + 2x^2y^2 + y^4$.
20. $(x^3 + y^3)^2 = x^6 + 2x^3y^3 + y^6$.
21. $(x^2 + y^2)^3 = x^6 + 3x^4y^2 + 3x^2y^4 + y^6$.
22. $(2a^2 + b^2)^2 = 4a^4 + 4a^2b^2 + b^4$.
23. $[(2a)^2 + b^2]^2 = 16a^4 + 8a^2b^2 + b^4$.
24. $$\begin{aligned}[(a + b)^2 + c^2]^2 &= (a + b)^4 + 2c^2(a + b)^2 + c^4 \\ &= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4 + \\ &\qquad 2c^2(a^2 + 2ab + b^2) + c^4 \\ &= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4 + \\ &\qquad 2a^2c^2 + 2abc^2 + 2b^2c^2 + c^4.\end{aligned}$$

Calculate:

25. $2 \cdot 3^2$.
26. $2^2 \cdot 3$.
27. $(2 \cdot 3)^2$.
28. $2 + 3^2$.

29. $(2 + 3)^2$. (Can you apply Problem 15 to this?)[8]

30. $2^2 + 3^2$.
31. $2(3 + 4)^2$.
32. $2 \cdot 3 + 4^2$.
33. $2 \cdot 3^2 + 4$.

34. $(2 + 3)^2 + 4 \cdot 5^2$. (Two ways.)

35. $2 + 3 \cdot 4 \cdot 5^2$.
36. $2 + 3 \cdot (4 \cdot 5)^2$.
37. $2 + (3 \cdot 4 \cdot 5)^2$.
38. $(2 + 3 \cdot 4 \cdot 5)^2$.
39. $(2 \cdot 3 + 4 \cdot 5)^2$.
40. $2 \cdot 3^2 + 4 \cdot 5^2$.
41. $(2 \cdot 3)^2 + (4 \cdot 5)^2$.
42. $(2 + 3)^2 \cdot (4 \cdot 5)^2$.

Multiply out:

43. $x(x + y)$.
44. $x(2x + 3y)$.
45. $xy(x + y)$.
46. $x^2(x + y)$.
47. $x^2(5x + 3y)$.
48. $x^2(x^2 + y^2)$.
49. $x^2y(x + y)$.
50. $x^2y^2(x + y)$.
51. $xy(x^2 + y^2)$.

52. $x^2y(x^2 + y^2)$.
53. $x^2y^2(x^2 + y^2)$.
54. $x(x^2 + 2x + 1)$.
55. $x(x^2 + 2xy + y^2)$.
56. $xy(x^2 + 2xy + y^2)$.
57. $x^2y^2(x^2 + 2xy + y^2)$.

58. $xy(x^3 + 3x^2y + 3xy^2 + y^3)$.
59. $x^2y^2(x^3 + 3x^2y + 3xy^2 + y^3)$.

60. $x^2y^2z^2(x^3y^2z + x^2y^3z^2)$.
61. $(xy)^2(x + y)$.
62. $(xy)^2(x^2 + y^2)$.
63. $(x^2)^2(x + y)$.
64. $(x^2y)^2(x + y)$.
65. $(x^2y^2)^2(x + y)$.
66. $(x^2y^2)^2(x^2 + y^2)$.
67. $(x^2)^3(x^2 + y^2)$.
68. $(x^3)^2(x^2 + y^2)$.
69. $(x^2y^2)^3(x^3 + y^3)$.
70. $(x^3y^3)^2(x^3 + y^3)$.

Take out the common factor:

71. $xy^2 + 2y^2$.
72. $x^2y^2 + 3y$.
73. $x^2y^2 + 3y^2$.
74. $xy^2 + x^2y$.
75. $x^2y + x^2y^2$.
76. $x^2y^2 + x^2y^2$.

77. $3x^2y^2 + 2x^2y^2$.
78. $2^5x^2y^6 + 2^6x^4y^3$.
79. $2^5x^2y^6 + 4^3x^4y^3$.
80. $4x^3 + 2x^2y$.
81. $4^2x^3y + 2^3x^2y^3$.
82. $x^3y^2 + x^2y^3$.
83. $xy + 3x + 4xy^2$.
84. $x^2y^2 + 5xy^2 + 4x^2y$.
85. $x(x + y) + y(x + y)$.
86. $x^2 + xy + xy + y^2$.
87. $x^2 + 2xy + y^2$.
88. $x(x + 2y) + y(x + 2y)$.
89. $x^2 + 3xy + 2y^2$.
90. $x^2(x + y)^2 + y^2(x + y)^2$.
91. $x^2y(x + y)^2 + xy^2(x + y)^2$.
92. $x^2y(x + y) + xy^2(x + y)^2$.

Frequently we have an expression whose value we wish to know when numerical values are given to the letters. For instance, $x^2 + y^2$, for $x = 2$ and $y = 3$, equals $4 + 9 = 13$.

Find the value of each of the following when $x = 2$ and $y = 3$:

93. $x(x^2 + y^2) + y(x^2 + y^2)$.
94. $xy(x^2 + y^2)$.
95. $(xy)^2(x + y)$.
96. $x^2y^2(x + y)$.
97. $(3x + 2y)(2x + 3y)$.
98. $x[x + y(x + y)]$.
99. $3x[2y + 2x(x + y)^2]$.
100. $(2x + y)^2(x + 2y)^2$.

5.2 Laws of Exponents

You probably have noticed that in a problem such as Problem 59 of our last exercise there seems to be a rule for multiplying two powers of the same base. If we have $x^2 \cdot x^3$, this means $(x \cdot x) \cdot (x \cdot x \cdot x) = x \cdot x \cdot x \cdot x \cdot x = x^5$. A little thought will convince you that this is a general rule. If we have x^n, which means the product of n x's, times x^m, which means the product of m x's, we shall have the product of $(m + n)$ x's. Our rule is:

5.2.1 *If m and n are natural numbers, then* $x^m \cdot x^n = x^{m+n}$.

Similarly, in Problem 67, there is the suggestion of the general rule which tells us how to raise a power to a power. In this problem we have $(x^2)^3$. This means, according to our definition of a power, $(x^2)(x^2)(x^2)$, which in turn means $(xx)(xx)(xx)$ or $x \cdot x \cdot x \cdot x \cdot x \cdot x = x^6$.

We see that this also is an example of a general rule:

5.2.2 *If m and n are natural numbers, then* $(x^m)^n = x^{m \cdot n} = x^{n \cdot m} = (x^n)^m$.

Consideration of problems like 61 leads us to the rule:

5.2.3 *If m is a natural number, then* $(xy)^m = x^m \cdot y^m$.

All three of these rules are easy to understand and remember if you understand and remember the definition of a power of a number, that is, if you remember that a^n is simply a shorthand way of writing the product of n a's.

Note: These rules are not postulates. They are *consequences* of our postulates, and we must accept them since we accepted the postulates. Thus, in the third rule, $(xy)^m$ means $(xy)(xy) \cdots (xy)$ where we have m factors (xy). By using the commutative and associative laws for multiplication of natural numbers, this may be written as

$$(x \cdot x \cdot \cdots \cdot x)(y \cdot y \cdot \cdots \cdot y),$$

where there are m x's in the first parentheses and m y's in the second parentheses, and this, by definition, is $x^m \cdot y^m$.

This last rule can be extended to more than two factors. Thus

$$(xyzw)^m = x^m y^m z^m w^m.$$

These three rules are generally called laws of exponents. Later we shall learn of some additional laws of exponents.

Rules such as these may be looked upon as methods for speeding up our work. We could get along without them but they are handy things to know about. The multiplication tables which you learned in grade school are also rules of this kind. You *could* multiply 3 by 5 by saying it equals $3 + 3 + 3 + 3 + 3$, and then adding. It is much handier and speedier to know that $5 \cdot 3 = 15$.

Examples 5.2.4

1. $a^3 \cdot a^5 = a^{3+5} = a^8$. **2.** $(x^3)^2 = x^{3 \cdot 2} = x^6$.

3. $(3s)^4 = 3^4 \cdot s^4 = 81s^4$. **4.** $(a^2b)^3 = (a^2)^3 \cdot b^3 = a^6b^3$.

5. $(s^2t)^2 \cdot (st^2)^2 = (s^4t^2)(s^2t^4) = s^6t^6 = (st)^6$.

6. $u^3(u^2 + uv + v^2) = u^5 + u^4v + u^3v^2$.

7. $x^2y[x + y^2(x + y)] = x^2y[x + xy^2 + y^3] = x^3y + x^3y^3 + x^2y^4$.

EXERCISE 5.2.5

Apply the laws of exponents:

1. $x^2\cdot x^5$.
2. $a^4\cdot a^2$.
3. $(a^4)^2$.
4. $a^5\cdot(a^3a^4)$.
5. $a^5a^3a^2$.
6. $p^{10}p^6p$.
7. $x\cdot x^2\cdot x^4\cdot x^8$.
8. $(s^2)^3$.
9. $(r^3)^2$.
10. $(ab)^5$.
11. $(a^2b)^5$.
12. $(2^3)^2$.
13. $(3^2x)^2$.
14. $(u^3v^2)^4$.
15. $(x^m)^2$.
16. $(y^2)^n$.
17. a^tb^{2t}.
18. $(x^3x^4)^2$. (Two ways.)
19. x^mx^{2m}. (Two ways.)
20. $(rs^2t)^2$.
21. $(a+b)(a+b)^3$.
22. $(x+2y)^2(x+y)^2(x+2y)(x+y)^3$.
23. $a^2x^3\cdot a^3x^2$.
24. $(by)^2\cdot b^2y$.
25. $(r^2s)^2\cdot(rs^2)^3$.
26. $(3p)^2(2p)^3$.
27. $(a^3b^2c)^2(ab^2c^3)^2$.
28. $(x^2y)^3(xy^2)^3(x^3y)^2$.
29. $(p^4q^2)(q^2r^3)$.
30. $(ab^2)^3(b^2c)^3$.
31. $[(x^2q)^2]^3$.
32. $(p^2q^3)^2[(pq^2)^3]^2$.

Multiply out:

33. $x^2(x^3+xy+y^3)$.
34. $a^3b(a^3+6a^2b+8ab^2+b^3)$.
35. $p^5q^4r^3(p^2q^3r^4+2pq^2r^5)$.
36. $(2x^2)^2(x^2+xy+y^2)$.
37. $(x^2y)^3[(xy^2)^2+(x^2y)^2]$.
38. $2a^2b(a^3b+ab^3)+ab^2(a^2b^2+3a^4)$.
39. $s^2t[s+t(s+t)]$. (Two ways.)
40. $(s^2t^3u^4)^2\{st+u[t^2(s^2+t^2)+s^3u]\}$.

When we want to take out the common factor of an expression such as $a^4b^2+a^3b^2c$ we must remember that this equals

$$\begin{aligned} a\cdot a\cdot a\cdot a\cdot b\cdot b+a\cdot a\cdot a\cdot b\cdot b\cdot c &= a(a\cdot a\cdot a\cdot b\cdot b+a\cdot a\cdot b\cdot b\cdot c)\\ &= a\cdot a(a\cdot a\cdot b\cdot b+a\cdot b\cdot b\cdot c)\\ &= a\cdot a\cdot a(a\cdot b\cdot b+b\cdot b\cdot c)\\ &= a^3b(a\cdot b+b\cdot c)\\ &= a^3b^2(a+c). \end{aligned}$$

We examine the expression to determine the lowest power of each factor which appears in the terms, and this power of this factor will be a common factor of the terms. Thus a^3 is the lowest power of a, b^2 is the lowest power of b, but c is not a factor at all (of the first term). Thus a^3b^2 is a common factor of the two terms. Of course, ab and a^2b and ab^2 and a^2b^2 are also common factors, but a^3b^2 is the *highest* common factor. After it is factored out, there remain no more common factors.

After we have found the highest common factor, we apply the distributive law by answering the question, "By what must we multiply the common factor in order to obtain each of the terms of the given expression?" For instance, if we have $x^3y^3 + xy^4$, the highest common factor is xy^3. Then xy^3 must be multiplied by x^2 to obtain x^3y^3 and by y to obtain xy^4. Then $x^3y^3 + xy^4 = xy^3(x^2 + y)$.

EXAMPLES 5.2.6

1. $ab^2 + a^2b = ab(b + a)$.
2. $x^4y^3z^2 + x^2y^3z^4 = x^2y^3z^2(x^2 + z^2)$.
3. $2^4a^2b^3c^2 + 2^6a^3b^4c = 2^4a^2b^3c(c + 2^2ab)$.
4. $r^3(r + s)^2 + r^2(r + s)^3 = r^2(r + s)^2[r + (r + s)] = r^2(r + s)^2(2r + s)$.

EXERCISE 5.2.7

Take out the (highest) common factor:

1. $x^2y + xy^2$.
2. $p^3r^2 + p^2r^3$.
3. $ab^2c^3 + a^3b^2c$.
4. $u^2v^3w^4 + u^4v^2w^3 + u^3v^4w^2$.
5. $3x^2y^5 + 5x^3y^3 + x^4y^3z$.
6. $2a^3b^2 + 4ab^3 + 8a^2b^4$.
7. $9pq^2r^2 + 27p^3q^2 + 3^4pq^3r^3$.
8. $15x^5y^3z^2 + 25x^2y^3z^2$.
9. $15p^4q^3 + 21p^7q^4r^2$.
10. $5^2st^5 + 3^2s^5t$.
11. $25ab^8c^3 + 75a^2b^5 + 125b^3c^3$.
12. $9x^2y^2 + 3^3x^3y^3$.
13. $2^7r^3s^2t^2 + 2^5r^5s^2t^3 + 4r^2st$.
14. $x^2(y + z)^3 + x^3(y + z)^2$.
15. $a^2(a + b)^3 + 3a^3(a + b)^2$.
16. $(u + v)(u^2 + v^2) + (u + v)^2(u^2 + v^2)$.
17. $(x + y)^2 + (x + y)^3$.
18. $(x^2 + 2xy + y^2) + (x + y)^3$.
19. $x(x + y) + y(x + y)$.

20. $(x^2 + xy) + y(x + y)$.
21. $(x^2 + xy) + (xy + y^2)$.
22. $x^2 + 2xy + y^2$.

Translate into equations:

23. If the sides of a square are each increased by 1 yard, the area becomes 16 square yards.

24. The sides of two squares differ by 1 foot. The sum of the areas of the squares is 25 square feet.

25. The sum of the squares of two consecutive natural numbers is 25.

26. The square of the sum of two consecutive natural numbers is 25.

27. Two consecutive natural numbers are such that the square of the first added to four times the square of the second gives the sum 169.

28. If the edges of a cube are each increased by 3 feet, the volume becomes 125 cubic feet.

29. The edges of two cubes differ by 1 foot. The volume of the smaller added to twice the volume of the larger is 155 cubic feet.

A famous rule of geometry states that when we have a right triangle (that is, one with one of its angles a right angle) the square of the longest side (called the hypotenuse) equals the sum of the squares of the shorter sides (called the legs). This is known as the Pythagorean theorem after a famous Greek mathematician, Pythagoras, who lived about 500 B.C.

30. If the sides of a right triangle are a, b, and c, with c the hypotenuse, state the Pythagorean theorem in the form of an equation.

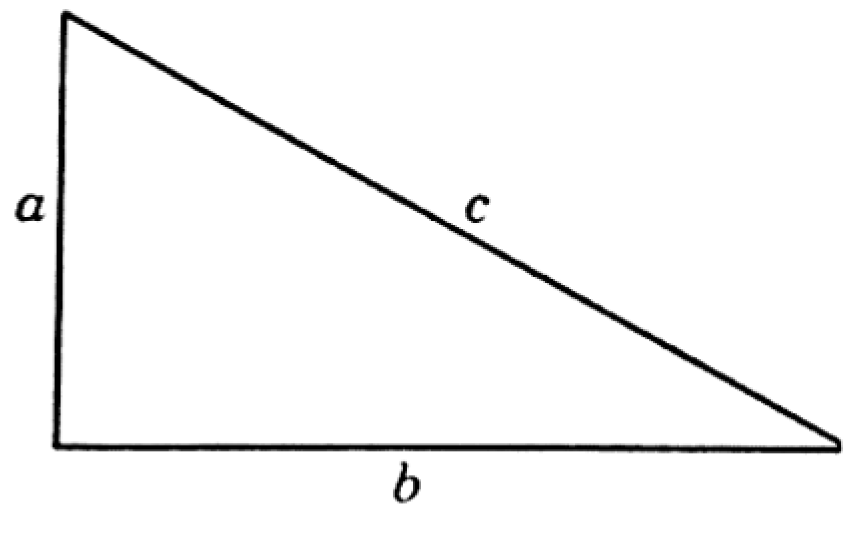


FIG. 5.2.8.

31. The two legs of a right triangle differ by 1 foot; the hypotenuse is 5 feet.

32. One of the legs of a right triangle is twice one more than the other leg; the hypotenuse is 13.

33. The lengths of the legs of a right triangle are consecutive natural numbers, and the hypotenuse is 29.

34. The hypotenuse of a right triangle is 8 feet more than one of the legs, and the other leg is 20 feet.

5.3 Like Terms

In some of the problems of Exercise 5.2.5, such as Problem 38, we run into the question of "like terms." By like terms we mean terms in which the literal factors (the letter factors) are the same and to the same powers. In Problem 38 we have

$$2a^2b(a^3b + ab^3) + ab^2(a^2b^2 + 3a^4) = 2a^5b^2 + 2a^3b^4 + a^3b^4 + 3a^5b^2.$$

The first and last terms on the right side of the = sign are "like terms" as are the second and third. We can shorten the expression by "combining like terms." Then this may be written as $5a^5b^2 + 3a^3b^4$.

What we have done has been to add the numerical factors of the like terms. This is merely an application of our distributive law. For we have

$$2a^5b^2 + 3a^5b^2 = a^5b^2(2 + 3) = 5a^5b^2$$

and

$$2a^3b^4 + a^3b^4 = a^3b^4(2 + 1) = 3a^3b^4.$$

The numerical factor of a term is called the *coefficient* of that term. The coefficient of the first term of Problem 5 of the last exercise is 3, of the second term is 5, and of the third term is 1. (When the coefficient is 1 we generally do not bother to write it at all.) Using this new word, we can state our rule as:

5.3.1 *Like terms may be combined by adding their coefficients.*

Exercise 5.3.2

Multiply out and combine like terms:

1. $a + b + 2(3a + 4b) + 3(2a + b)$.

2. $3p + 4p + 2q + 3[p + 2(p + q)]$.

3. $s(t + u) + 2u(s + t) + 3t(u + s)$.

4. $3x(x + y) + 2y(x + y)$.
5. $(3x + 2y)(x + y)$.
6. $a^2(b + c) + b^2(c + a)$.
7. $3p^2(p + q) + 2pq(p + 2q) + q^2(2p + q)$.
8. $3 + 2(a + 2b) + 4(2a + 3)$.
9. $7(p + 3q) + 5(p + 2) + 4(q + 3)$.
10. $3x^2 + 2[x + 2x(x + 4)]$.
11. $r^3[s^2 + r(s + t)] + 2r^2[r^2s + r(rt + s^2)]$.
12. $2 + 3\{a + k + 2[3 + 3a + 2k + 3(a + k)] + a\}$.
13. $2 + 3(a + k) + 2[3 + a + 2(k + 2) + 3(a + k) + 1]$.
14. $(a + b)(a + c) + 2a(b + c) + 3b(a + c)$.
15. $a^2(b + c) + b^2(c + a) + c^2(a + b)$.
16. $2x^4[x + y^2(x + z^2) + 3z^2(x + y^2)] + 5x^5$.
17. $2p[3p^3 + 2p^2 + 2p + 4 + p(p^2 + 3p + 1)]$.
18. $r\{4 + st[(r + s^2 + 2t) + t(r + 3s + 2s^2)]\} + 5rst$.
19. $a^5(a^2b + ab^2) + a^4(a^3b + ab^3) + a^3(a^4b + ab^4) + a^2(a^5b + ab^5) + a(a^6b + ab^6)$.
20. $x^4y^4(x^2y + xy^2) + 2x^3y^3(x^3y^2 + x^2y^3) + 3x^2y^2(x^4y^3 + x^3y^4) + 4xy(x^5y^4 + x^4y^5) + 5(x^6y^5 + x^5y^6)$.
21. $x(x + 2) + 2(x^2 + x + 1)$.
22. $x^2(x + 4) + x(x^3 + 3x + 1)$.
23. $x^3(x + 3) + x^2(x^2 + 3x + 2) + x(x^3 + x^2 + 3)$.
24. $x^3(x^2 + 2x + 1) + x^2(x^3 + 3x^2 + 3x + 1) + x(x^4 + 4x^3 + 6x^2 + 4x + 1)$.
25. $x[x^2(x + 2) + x(x^2 + 4x + 4)] + x^2[x(x + 1) + (x + 1)]$.
26. $3x(x^2 + 6x + 9) + 9x^2(x + 3)$.
27. $2x^3[(x + 2) + 2(x^2 + 4)] + 4x^2[(x^2 + 6) + 4(x + 6)]$.
28. $x^2\{1 + x^2[1 + x^2(1 + x^2)]\}$.
29. $x(x + 1)^2 + x^2(x + 1)$.
30. $x(x + 1)(x + 2) + x(x + 2)(x + 3)$.

6

Subtraction of Natural Numbers

6.1 The Inverse Operations

We have spoken of the four arithmetical operations, addition, multiplication, subtraction, and division. Thus far we have discussed only addition and multiplication as applied to algebra. You may have begun to think (or hope) that we had forgotten about subtraction and division.

The operations of addition and multiplication are sometimes referred to as the direct operations. The operations of subtraction and division are called the inverse operations. Subtraction is said to be the inverse of addition, and division is said to be the inverse of multiplication. Subtraction is the inverse of addition because subtraction undoes addition. For instance, if we add 3 to 5 and then subtract 3 from the sum, we obtain the original number 5. Similarly division undoes multiplication for, if we multiply 5 by 3 and then divide the result by 3, we obtain the original 5. In symbols,

$$(5 + 3) - 3 = 5$$

and

$$(3 \cdot 5) \div 3 = 5.$$

(You, of course, remember the symbols for subtraction and division from arithmetic.)

Just as we started you in the study of addition and multiplication with some review problems from arithmetic, suppose that we look at some subtraction problems from arithmetic.

EXERCISE 6.1.1

Subtract the lower number from the upper:

1. $\begin{array}{r} 17 \\ \underline{8} \end{array}$
2. $\begin{array}{r} 27 \\ \underline{14} \end{array}$
3. $\begin{array}{r} 25 \\ \underline{17} \end{array}$
4. $\begin{array}{r} 73 \\ \underline{67} \end{array}$

5. $\begin{array}{r}56\\ \underline{24}\end{array}$ **6.** $\begin{array}{r}34\\ \underline{19}\end{array}$ **7.** $\begin{array}{r}128\\ \underline{67}\end{array}$ **8.** $\begin{array}{r}135\\ 18\end{array}$

9. $\begin{array}{r}555\\ \underline{164}\end{array}$ **10.** $\begin{array}{r}431\\ \underline{256}\end{array}$ **11.** $\begin{array}{r}642\\ 298\end{array}$ **12.** $\begin{array}{r}1776\\ \underline{652}\end{array}$

13. $\begin{array}{r}1812\\ \underline{1492}\end{array}$ **14.** $\begin{array}{r}2544\\ \underline{976}\end{array}$ **15.** $\begin{array}{r}6828\\ \underline{3909}\end{array}$ **16.** $\begin{array}{r}7002\\ \underline{567}\end{array}$

17. $\begin{array}{r}6000\\ \underline{3217}\end{array}$ **18.** $\begin{array}{r}63{,}428\\ \underline{22{,}643}\end{array}$ **19.** $\begin{array}{r}481{,}372\\ \underline{281{,}666}\end{array}$ **20.** $\begin{array}{r}4{,}392{,}763\\ \underline{2{,}804{,}868}\end{array}$

6.2 Subtraction

In working the subtraction problems, for instance the first, most of you probably said to yourselves, "8 from 17 is 9," and wrote down the 9. A few of you may have said, "8 and 9 are 17," and have written down the 9. Whichever way you did it, you were using your knowledge of addition tables to answer the question, "What do I need to add to 8 to obtain 17?"

If you are in a store, make a 32-cent purchase, and hand the clerk a dollar bill, how does the clerk make change? Generally, he says "32," counts out three pennies into your hand, saying at the same time "35," then places a nickel in your hand, saying "40," then a dime with the word "50," and finally a 50-cent piece (or two quarters), saying, "One dollar." If you were to ask the clerk how much change he had given you, probably he would not know! But he has given you the correct change by mechanically answering the question, "32 plus what makes 100?" He went through the same process when he was getting the change from the register. Watch the clerk getting change from the register the next time that you are in a store.

This illustrates the process of subtraction. To subtract a number from another number means to find the number which must be added to the first to obtain the second. To subtract b from a means to find the number x which when added to b gives us a; x is called the difference between a and b.

6.2.1 DEFINITION: If a and b are natural numbers, the difference between a and b, in that order, means a natural number x such that $b + x = a$. The difference is written $a - b$, so that $x = a - b$. To subtract b from a means to find $a - b$; $a - b$ may also be read "a minus b"; a is called the *minuend*, and b is called the *subtrahend.*

Examples 6.2.2

1. $5 - 3$ means a natural number x such that $3 + x = 5$. Of course, $x = 2$ in this case; that is, $5 - 3 = x = 2$.

2. $52 - 7$ means a natural number x such that $7 + x = 52$. Here $x = 45$, or $52 - 7 = 45$.

3. $738 - 127$ means a natural number x such that $127 + x = 738$. What is this x?

4. $11 - 17$ means a natural number x such that $17 + x = 11$. In this example *there is no such natural number.*

Recall that the set of natural numbers are the numbers 1, 2, 3, 4, · · ·. If we know our addition tables, we know that not one of these numbers is such that when added to 17 it will give the sum 11.

[Some of you may have heard of other kinds of numbers called "negative numbers," and you may want to say that we *can* subtract 17 from 11, getting the answer "minus 6." But we agreed to abide by the rules of our game. One of our rules was that we were to use the natural numbers only. We must not, unless and until we decide to change the game, use any other kind of number.]

Also recall what we mean when we say that a set of numbers is closed under an operation. We mean that, if we perform the operation on *any* numbers of the set, we shall obtain a number of the set. The set of natural numbers is closed under addition and multiplication. When we add two natural numbers the sum is a natural number; when we multiply two natural numbers the product is a natural number. But if we subtract a natural number from a natural number we may or may not get a natural number. The set of natural numbers is *not* closed under subtraction.

6.3 "Greater Than" and "Less Than"

In those cases in which the difference between a and b is a natural number, we say that a is greater than b or that b is less than a. We write $a > b$ and read this "a is greater than b," or we write $b < a$ and read this "b is less than a." If $a > b$ and $a - b = c$, we say that "a is c more than b." *

EXAMPLES 6.3.1

1. 9 is 4 more than 5, since $5 + 4 = 9$ or $9 - 5 = 4$.
2. 8 is 7 less than 15, since $8 + 7 = 15$ or $15 - 8 = 7$.
3. 18 is 9 more than 9, since $9 + 9 = 18$ or $18 - 9 = 9$.
4. 7 is 7 less than 14, since $7 + 7 = 14$ or $14 - 7 = 7$.
5. $17 - 9 = 8$ may be translated "17 is 9 more than 8" or "8 is 9 less than 17."
6. $11 + 8 = 19$ may be translated "19 is 8 more than 11" or "11 is 8 less than 19."

Since $11 + 8 = 8 + 11 = 19$, we may also say "19 is 11 more than 8" or "8 is 11 less than 19."

EXERCISE 6.3.2

Read each of the following:

1. If $x > y$, then $y < x$.
2. $17 > 11$.
3. If a and b are unequal natural numbers, then $a^2 + b^2 > 2ab$.
4. If $a > b$, then $(a - b)^2 + (a + b)^2 = 2(a^2 + b^2)$.
5. If p and q are unequal natural numbers, then $2pq < p^2 + q^2$.
6. $(a - b)^2 = (a^2 + b^2) - 2ab$.
7. $x^2 + (y - z)^2 = (x^2 + y^2 + z^2) - (2yz)$.

We have become accustomed to our agreement (4.2.1) that in any problem involving both addition and multiplication the multiplications are to precede the additions unless parentheses indicate otherwise. We make the additional agreement that

6.3.3 *In a problem involving both subtraction and multiplication the multiplications are to precede the subtractions unless parentheses indicate otherwise.*

---

* The symbols for greater than (>) and less than (<) were first used by Thomas Harriot. They appeared in his posthumous publication *Artis analyticae praxis* (1631). It is interesting to note that musicians use these same symbols in almost the same sense as mathematicians. To a musician > means "diminuendo," that is, go from a loud tone to a soft tone, and < means "crescendo," that is, go from a soft tone to a loud tone. The earliest known use of these signs by musicians is in Geminiani's *Prime Sonate* (1739).

EXERCISE 6.3.2 (continued)

8. $(x + y)^2 - 2xy = x^2 + y^2$.
9. $a^2 - b^2 = (a + b)(a - b)$.
10. $p^2 - (q + r)^2 = [p - (q + r)][p + q + r]$.
11. $(p + q)^2 - r^2 = [(p + q) - r][p + q + r]$.
12. $(2r - s)^2 = (4r^2 + s^2) - 4rs$.
13. $(r - 2s)^2 = (r^2 + 4s^2) - 4rs$.

Translate each of the following into two statements, one using the phrase "more than" and the other the phrase "less than":

14. $21 - 6 = 15$.
15. $75 - 21 = 54$.
16. $569 - 378 = 191$.
17. Do this for Problems 6 through 13 above.
18. Explain what is meant by the statement $57 - 29 = 28$ in terms of Definition 6.2.1.
19. Which of the following are meaningless in terms of natural numbers? Explain.

(*a*) $95 - 37$.
(*b*) $37 - 95$.
(*c*) $8 - 8$.
(*d*) $163 - 79$.
(*e*) $563 - 564$.
(*f*) $7 - 3\cdot 2$.
(*g*) $9 - 5\cdot 2$.
(*h*) $7\cdot 5 - 5\cdot 7$.

Calculate:

20. $3\cdot 7 - 8$.
21. $26 - 6\cdot 4$.
22. $7 + (9 - 3)$.
23. $7 - (9 - 3)$.
24. $(19 - 3) - 7$.
25. $(6 + 17) - 8$.
26. $(16 - 3) + (27 - 9)$.
27. $3 + (17 - 8) + (53 - 27)$.
28. $54 - [17 + (25 - 22)]$.
29. $(38 - 27) - (59 - 52)$.

30. $[(67 - 20) - (89 - 63)] + 5$.
31. $(85 - 39) - [98 - (77 - 24)]$.
32. $[100 - (93 - 42)] + [73 - (164 - 119)]$.
33. $[100 - (93 - 42)] - [73 - (164 - 119)]$.

34. $37 - (17 - 8)$.
35. $(37 - 17) - 8$.
36. $(37 - 17) + 8$.
37. $37 - (17 + 8)$.

Evaluate for the indicated values of the letters:

38. $a^2 - b^2$; $a = 5$, $b = 3$.
39. $(a + b)^2 - (a - b)^2$; $a = 4$, $b = 2$.
40. $a - (a - b)$; $a = 6$, $b = 2$.
41. $x(x - y) + y(x - y)$; $x = 5$, $y = 3$.
42. $(x - y)(x + y)$; $x = 5$, $y = 3$.
43. $x^2 - y^2$; $x = 5$, $y = 3$.
44. $x^3 - y^3$; $x = 7$, $y = 4$.
45. $(x - y)(x^2 + xy + y^2)$; $x = 7$, $y = 4$.
46. $x^3 + y^3$; $x = 2$, $y = 5$.
47. $(x + y)[(x^2 + y^2) - xy]$; $x = 2$, $y = 5$.
48. $x^4 - y^4$; $x = 3$, $y = 2$.
49. $(x^2 - y^2)(x^2 + y^2)$; $x = 3$, $y = 2$.
50. $(x - y)(x + y)(x^2 + y^2)$; $x = 3$, $y = 2$.
51. $(s^2 + t^2)[s^4 - (s^2t^2 - t^4)]$; $s = 2$, $t = 1$.
52. $s^6 + t^6$; $s = 2$, $t = 1$.
53. $(s^2 - t^2)(s^4 + s^2t^2 + t^4)$; $s = 2$, $t = 1$.
54. $(s - t)(s + t)[(s^2 + t^2) - st][s^2 + t^2 + st]$; $s = 2$, $t = 1$.
55. $s^6 - t^6$; $s = 2$, $t = 1$.

6.4 Equivalent Equations

Example 6.4.1

Referring to Problem 1 of Exercise 3.1.2 which reads, "Seven more than a certain number is 15," you should have obtained the equation $x + 7 = 15$. It is not difficult to guess that $x = 8$ is the solution. But, using the subtraction notation, this problem may be translated into $15 - x = 7$. Again obviously, $x = 8$ is the solution. Two such equations, that is, two equations which have the same solutions, are said to be *equivalent* equations. Also, since $x + 7 = 7 + x$, the first equation may be written

$$7 + x = 15$$

and, from our definition of subtraction,

$$x = 15 - 7 = 8.$$

Jammu & Kashmir University Library,
Accession No ....

EXERCISE 6.4.2

Translate into equations (if possible, obtain two equivalent equations):

1. 43 is one more than twice a certain number.

2. 56 is a certain number more than three times that same number.

3. One number is three less than another and their sum is 33.

4. Two numbers are such that one is three less than twice the other; their sum is 21.

5. The first of three numbers is 4 less than the second, and the third equals the sum of the first two; the sum of the three numbers is 40.

6. The difference between the squares of two consecutive natural numbers is 9.

7. One of the legs and the hypotenuse of a right triangle are consecutive natural numbers; the other leg is 3.

8. The hypotenuse of a right triangle is one more than one of the legs; the other leg is 5.

9. One of the legs of a right triangle is 8 less than the hypotenuse, and the other leg is 20.

10. Michael has 15 cents more than Albert, and together they have 39 cents.

11. Russell had three times as much money as Paul. But Russell lost 16 cents and then had only as much as Paul.

12. In 5 years Dick will lack 4 years of being twice as old as he is now.

13. The perimeter of a rectangle is 70 and the length is 9 more than the width.

14. Joe is twice as old as John, and the difference of their ages is 14 years. (Compare with Problem 5, Exercise 3.1.2.)

15. A piece of cord 42 feet long is cut so that the difference of the two lengths is 14 feet.

16. A cord 59 inches long is cut into three pieces of which the first is twice as long as the second and the first is also 4 inches shorter than the third.

17. A father earns twice as much each day as his son. The father works 8 days, and the son works 5 days. The difference between their earnings is $44.

18. Three baseball players, Larry, Luke, and Al, made 44 hits in 11 games. Larry had 2 more than Luke and 4 less than Al.

19. By July 4, the sixty-third day of the baseball season, the Cleveland Indians, the New York Yankees, and the Boston Red Sox had each played a total of 68 games. The Indians had won 4 fewer than the Red Sox and 6 more than the Yankees. The total number won by all three teams was 142.

Solve the following equations, using the definition of subtraction:

20. $11 + x = 29$. **21.** $x + 11 = 29$.
22. $x + 7 = 30$. **23.** $m + 29 = 51$.

24. $s + [3(5 + 4) + 9] = 63$.
25. $y + 1 + 4[3 + 7(8 + 1)] = [398 + 5(4 + 2)]$.

Set up the equations and solve:

26. If a certain number is added to 11, the sum is 29.
27. Albert has 51 marbles, which is 29 more than James has.

7

Zero and the Natural Numbers

7.1 Zero

In all our work thus far we have been using the natural numbers only. Indeed, we have repeatedly emphasized this fact.

This has led to some inconveniences. Whereas the set of natural numbers, that is, the numbers 1, 2, 3, 4, $\cdots$, is closed under the operations of addition and multiplication, we discovered that it is not closed under the operation of subtraction. When we subtract a smaller natural number from a larger we obtain a natural number; but, if we try to subtract a larger number from a smaller or if we try to subtract a number from itself, we do not obtain a natural number. For instance, there is no natural number x such that $7 + x = 3$, or $3 - 7$ is not a natural number, and there is no natural number x such that $5 + x = 5$, or $5 - 5$ is not a natural number.

Recall that we are looking upon algebra as a game. If we believe we can improve the game by adding to the rules or by introducing new equipment, we are free to do so. However, we must always remember to play according to the rules upon which we agree and with the equipment which we agree to use. We must always know what game we are playing and what our equipment is. Chess and checkers are played on the same kind of board. But they are quite different games, played with different pieces and with different rules. We must not make chess moves if we are playing checkers, and we must not make checker moves if we are playing chess, and we must play chess with chess men and checkers with checkers.

After this introduction you may have fears that we are about to throw away the whole game which we have built up. Actually we are not going to discard any of it. We are going to add to it by introducing some new equipment (numbers) and also by adding some new moves (operations).

The first new number that we introduce is the number *zero.* We do this, from our present point of view, so that we shall have an "answer" when we subtract a number from itself.

7.1.1 Definition: The number zero, symbolized by 0, shall be the number which has the property $a + 0 = a$ or $a - a = 0$.

Zero is the number which when added to another number gives a sum equal to that other number; or zero is the number obtained when a number is subtracted from itself. Example: $5 + 0 = 5$; $5 - 5 = 0$.*

We have said that we are not going to discard any rules of our old game. All the rules which we have built up are going to apply to this new number. Thus,

$$\begin{aligned} a + 0 &= 0 + a, \\ a + (b + 0) &= (a + b) + 0, \\ a \cdot 0 &= 0 \cdot a, \\ a \cdot (b \cdot 0) &= (ab) \cdot 0, \\ a(b + 0) &= ab + a \cdot 0, \\ 0(b + c) &= 0 \cdot b + 0 \cdot c. \end{aligned}$$

When we add to our system (that is, introduce new equipment) in such a way that the rules which hold for the original system continue to hold in the new (larger) system, we say that we have *generalized.* This is a process familiar to mathematicians. Here we have generalized by throwing in the number zero with our natural numbers. Our new system is the set of numbers

$$0, 1, 2, 3, 4, \cdots.$$

---

* We have, of course, quite regularly been using the symbol 0. Back in Problem 12 of Exercise 2.1.1 we had the number 809 which makes use of the symbol 0. We have assumed that you knew what this meant. Here the symbol 0 is used to indicate the fact that we are talking about the number eight hundred nine. If we omitted it the number would read 89. Later we shall discuss more fully this use of 0. We shall discover that it ties in nicely with our present use. See page 186.

Recall that, if b and c are natural numbers, then $b \cdot c = c + c + \cdots + c$ where there are b c's in the sum. By our generalization we want

$$b \cdot 0 = 0 + 0 + \cdots + 0$$

where there are b zeros in the sum. Then

$$a + b \cdot 0 = a + (0 + 0 + \cdots + 0)$$

where again there are b zeros in the sum. (Recall the meaning of the = sign, page 7.) But, if 0 is to obey all the rules which our original system (the natural numbers) obey, $a + (0 + 0 + \cdots + 0)$ can be written with symbols of grouping in any way that makes sense. In particular,

$$a + (0 + 0 + \cdots + 0) = \{[(a + 0) + 0] + 0\} + \cdots + 0$$

where we have grouped the first two terms, then this group is grouped with the third term, then this with the fourth term, and so on until we have grouped all the terms. The first grouping (parentheses) is by definition equal to a. Then the second grouping (brackets) equals $[a + 0]$, and this is again a. Finally, we see that the right side is simply a itself. Therefore

$$a + b \cdot 0 = a + 0 + 0 + \cdots + 0 = a.$$

But the number which when added to a gives a as the sum is this new number 0. We therefore see that for any natural number b

7.1.2 $$b \cdot 0 = 0.$$

We want 0 itself to obey this new rule, and therefore also

$$0 \cdot 0 = 0.$$

Examples 7.1.3

1. $532 + 0 = 532.$
2. $674 - 674 = 0.$
3. $17(9 - 9) = 17 \cdot 0 = 0.$
4. $63 + [5(19 - 19)] = 63.$

A nice question is, what do we obtain when we subtract 0 from a number, that is, what is $a - 0$? By definition of subtraction (which we, of course, want to apply to this new number zero), $a - 0$ is a number x (of our enlarged system) such that

$$0 + x = a$$

or

$$a - 0 = x.$$

But

$$0 + x = x + 0 = x \quad \text{(by definition of zero)}$$

and therefore

$$a - 0 = a;$$

and also

$$0 - 0 = 0.$$

5. $7 - 0 = 7$.

If the difference between two numbers (of our new system) is a natural number, we say that the first number is greater than the second or the second is less than the first. (We said exactly the same thing about our original system.) Thus,

$$7 > 0 \quad \text{or} \quad 0 < 7;$$

and in general, if a is a number of our system different from zero,

$$a > 0 \quad \text{or} \quad 0 < a.$$

Since zero is the smallest number of our new system, $0 - a$, where a is not zero, must, for the present, remain meaningless.

6. $0 - 5$ is not a number of our system since there is no number x of our system such that

$$5 + x = 0.$$

From our definition of the difference between two numbers, if we have the equation

$$y - 4 = 7,$$

we obtain

$$y = 7 + 4 = 11.$$

This suggests a method for solving equations of the type

$$x - a = b.$$

For then

$$x = b + a.$$

In particular, if we have the equation

$$x - 3 = 0,$$

$$x = 3 + 0 = 3.$$

A rule which will be of considerable use to us later is

7.1.4 POSTULATE: If a is not equal to zero and b is not equal to zero, then $a \cdot b$ is not equal to zero.

It is convenient at times to have a symbol for "not equal to." The symbol used is $\neq$, which is simply an equals sign crossed out. Using this, our postulate may be written:

7.1.4 *If* $a \neq 0$ *and* $b \neq 0$, *then* $a \cdot b \neq 0$.

The importance of this postulate lies chiefly in a conclusion we can draw from it. This is:

7.1.5 *If* $ab = 0$, *then either* $a = 0$ *or* $b = 0$ *or both equal zero.*

This follows at once from our postulate since the only other possibility is that neither a nor b equals zero in which case the product would not be zero. More generally:

7.1.6 *If the product of any number of numbers equals zero, then at least one of the numbers is zero.*

An application of this rule is its use in solving equations such as

$$(t - 1)(t - 3) = 0.$$

For then either $t - 1 = 0$ or $t - 3 = 0$ or both, and then, by the definition of the difference of two numbers, either $t = 1 + 0 = 1$ or $t = 3 + 0 = 3$ or both.*

---

* If at this time we seek to verify that $t = 1$ and $t = 3$ are actually solutions of the given equation, we meet a difficulty. For $t = 3$ we have, on replacing t by 3 in the given equation, $(3 - 1)(3 - 3) = 2 \cdot 0 = 0$, and we see that 3 is a solution. But for $t = 1$ we have $(1 - 1)(1 - 3)$ on the left of the equals sign, and the second factor of this is not part of our equipment.

We have seen that $0 \cdot b = 0$ if b is a natural number or zero. However $(1 - 3)$ is not a natural number or zero. This difficulty will be removed very shortly when we invent a new kind of number and add it to our equipment. For the present we shall be satisfied if we succeed in emphasizing the statement, "if the product of two numbers is zero then at least one of the numbers must be zero."

Note: This rule holds *only* when the product is *zero*. For instance, it is not correct to say that, if $(t - 1)(t - 3) = 4$, then either $t - 1 = 4$ or $t - 3 = 4$. For then we should conclude, from the definition of subtraction, that either $t = 4 + 1 = 5$ or $t = 4 + 3 = 7$. Neither of these values of t make

$$(t - 1)(t - 3) = 4.$$

Note: Some students form the habit of reading the symbol 0 as "nothing." We do not consider this good. The symbol 0 is to be read "zero." "Nothing" is that which does not exist. A square circle or a four-sided triangle or a three-legged animal with a green tail and pink ears are each nothing. Zero is a perfectly respectable number, having all the properties (except one, which we shall discuss later) of other numbers. In fact it has some properties which other numbers do not have, such as $a \cdot 0 = 0$ for any number a.

We have used notation for "greater than," $>$, and "less than," $<$. At times we wish to say that one number is *greater than or equal* to another number. The symbol we use for this is $\geqq$. Similarly, we read $\leqq$ as "less than or equal to."

$x^2 + y^2 \geqq 0$ is read "x^2 plus y^2 is greater than or equal to 0." For what values of x and y does the $=$ sign hold?

The symbol $\geqq$ may also be read "is not less than." Thus $5 \geqq 2$ may be read "5 is not less than 2" and $7 \geqq 7$ may be read "7 is not less than 7." $x^2 + y^2 \geqq 0$ may be read "$x^2 + y^2$ is not less than zero." Similarly $\leqq$ may be read "is not greater than."

$$2 < x \leqq 5$$

is read "x is greater than 2 but less than or equal to 5" or "x is greater than 2 but not greater than 5."

EXERCISE 7.1.7

Read each of the following:

1. $9 - 5 > 0$.
2. $0 < 25 - (18 - 1)$.
3. If $x > 0$ or $y > 0$ or both, then $x^2 + y^2 > 0$.
4. $a^2 + b^2 \geqq 2ab$.
5. $a^2 + b^2 \geqq 0$.
6. $(a + b)^2 \geqq 0$.
7. If $a \geqq b$, then $(a - b)^2 \geqq 0$.
8. $(a + b)^2 - (a^2 + 2ab + b^2) = 0$.
9. If $(x + 2)(x + 3) = 0$, then either $x + 2 = 0$ or $x + 3 = 0$ or both.

10. If $(y - 3)(y - 5) = 0$, then either $y - 3 = 0$ or $y - 5 = 0$ or both.

11. If $(x - 1)(x - 2)(x - 3) = 0$, then at least one of $x - 1 = 0$ or $x - 2 = 0$ or $x - 3 = 0$, or $x = 1$ or 2 or 3.

12. If $(p - 1)(p - 2)(p - 3)(p - 4) = 0$, then at least one of $p - 1 = 0$, $p - 2 = 0$, $p - 3 = 0$, or $p - 4 = 0$ or $p = 1$, $p = 2$, $p = 3$, or $p = 4$.

13. $4 \leqq x < 7$.
14. $4 < x \leqq 7$.
15. $4 \leqq x \leqq 7$.
16. $4 < x < 7$.

Express each of the following in symbols:

17. x is greater than 4 but less than 7.
18. x is greater than 0 but not greater than 3.
19. x is not less than 0 but is less than 3.
20. x is less than 8 but not less than 3.
21. x is not less than 1 and not greater than 15.
22. x is between 1 and 2.
23. x is between 5 and 0.

Calculate:

24. $18 - [7 - (3 - 3)]$.
25. $18 - [(7 - 3) - 3]$.
26. $18 - [(7 - 3) + 3]$.
27. $18 - [7 + (3 - 3)]$.
28. $18 - [(7 + 3) - 3]$.
29. $18 - [(7 + 3) + 3]$.
30. $18 - [7 + (3 + 3)]$.
31. $23 - [74 - (67 - 16)]$.

32. $\{98 - [74 - 3(30 - 7)]\} - \{57 + 7[33 - 3(18 - 7)]\}$.
33. $2(75 - 73)^3 + \{85 - [34 + 3(21 - 4)]\}^4$.

Solve the equations:

34. $t - 7 = 4$.
35. $t - 11 = 0$.
36. $y - 7 = 0$.
37. $x - 9 = 1$.
38. $x - 10 = 0$.
39. $x - 13 = 0$.
40. $x - 1 = 12$.
41. $s - 52 = 7$.
42. $s - 59 = 0$.
43. $(p - 3)(p - 6) = 0$.
44. $(x - 5)(x - 11) = 0$.
45. $(r - 18)(r - 21) = 0$.
46. $(m - 5)(m - 7) = 0$.
47. $(x - 2)(y - 4) = 0$.

48. $(x - 5)(x - 11)(x - 13) = 0$. Is the equation of Problem 48 equivalent to that of Problem 44?[9]

49. $(x - 5)^2(x - 11) = 0$.

50. $(x - 5)(x - 11)^2 = 0.$
51. $(x - 5)^2(x - 11)^2 = 0.$
52. $(x - 5)^2(x - 11)^2(x - 13)^2 = 0.$
53. $(x - 3) - 2 = 4.$
54. $[(x - 5) - 2] - 6 = 3.$
55. $[(y - 1) - 5] - 7 = 0.$
56. $[(t - 1) - 2][(t - 2) - 3] = 0.$
57. $\{[(s - 3) - 2] - 5\}\{[(s - 7) - 2] - 6\} = 0.$
58. $[(s - 4) - 6][(s - 6) - 9] = 0.$
59. $x - [(7 - 3) + (8 - 5)]^2 = 0.$
60. $x - \{99 - [77 - (83 - 65)]\} = 0.$
61. $t - \{\overline{[89 - (98 - 95)^4] - 8} + \overline{39 - (73 - 67)^2}\} = 0.$

Translate *and find the unknown number:*

62. Seven less than a certain number is 4.
63. Eleven less than a certain number is zero.
64. The difference between a certain number and 9 is one.
65. The difference between a certain number and 13 is zero.
66. When 52 is subtracted from a number we get 7.
67. When 59 is subtracted from a number we get zero.

68. The product of 3 less than a certain number and 6 less than this same number is zero.

69. If the difference between a certain number and 18 is multiplied by the difference between this number and 21, we get the product zero.

70. If 5 and 7 are each individually subtracted from a certain number, the product of the two differences is zero.

71. If 5, 11, and 13 are each individually subtracted from a certain number, the product of the three differences is zero.

72. James has 7 fewer marbles than Albert. James has 23 marbles. How many does Albert have?

73. Russell has 6 more marbles than Paul, and Paul has 10 more than Robert. The number Russell has is 49. How many do Paul and Robert each have?

74. The number of games played by each team of the American League during the regular baseball season is 154. During the 1950 season the Yankees won one more game than the Red Sox, the Red Sox won two more than the Tigers, and the Tigers won five more than the Indians. The Yankees won 93 games. How

many games did the Red Sox, the Tigers, and the Indians each win?

75. Is the set of numbers 0, 1, 2, 3, 4, ··· closed under addition?

76. Is the set of numbers 0, 1, 2, 3, 4, ··· closed under multiplication?

77. Is the set of numbers 0, 1, 2, 3, 4, ··· closed under subtraction?

8

The Integers; Addition and Subtraction

8.1 The Positive and Negative Integers

We have pointed out that $0 - b$, where b is not zero, is not a number of our system:

$$0, 1, 2, 3, 4, \cdots$$

and therefore $0 - b$ is, for the moment, without meaning. Indeed, the symbol $a - b$ is, for the moment, without meaning if $a < b$ and a and b are any numbers of our system.

When we placed the number 0 in with our set of natural numbers we gave the symbol $a - a$ a meaning. We shall presently introduce more new numbers which will give the symbol $a - b$ meaning in all cases. These new numbers will obey all the rules of our old numbers; we generalize again. This time the generalization will be such that our new set of numbers will be closed under subtraction.

Examples 8.1.1

1. If the temperature in the evening is 10 degrees *above* zero we generally indicate this by $+10$. If during the night the temperature drops 15 degrees, what is the temperature in the morning? It is, of course, 5 degrees *below* zero. We generally indicate this by -5.

2. Mount Whitney is 14,495 feet *above* sea level. We might indicate this by $+14{,}495$. Death Valley is 280 feet *below* sea level. We might indicate this by -280.

3. A man has assets worth \$6000; we might indicate this by $+6000$. He has debts of \$2000; we might indicate this by -2000. (Or we might write the 6000 in black and the 2000 in red.)

4. The latitude of Kent, Ohio, is 41° north; we might indicate this by $+41$. The latitude of Buenos Aires, Argentina, is 35° south; we might indicate this by -35.

5. The longitude of Kent, Ohio, is 82° west; we might indicate this by $+82$. The longitude of Bombay, India, is 73° east; we might indicate this by -73.

6. If a wheel rotates in a counterclockwise direction through 10 revolutions, we might indicate this by $+10$. If it rotates in a clockwise direction through 17 revolutions, we might indicate this by -17.

7. If a certain manufactured article is supposed to weigh 25 pounds but weighs 25 pounds and 7 ounces, we might indicate this by $+7$. If it weighs 24 pounds and 6 ounces we might indicate this by -10. (How might we indicate that it weighs 26 pounds?)

8. If a baseball team wins 93 games, we might indicate this by $+93$. If it loses 61 games, we might indicate this by -61.

9. If a point 5 feet to the right of a certain fixed point is indicated by $+5$, then we might indicate the point 3 feet to the left of this fixed point by -3.

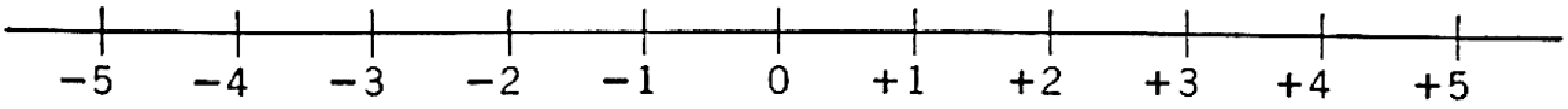


FIG. 8.1.2.

In each of these statements the numbers with the + or − signs in front of them indicate not only a quantity but also another property. We might say that the properties of those with the + signs are opposite to the corresponding ones with the − signs. In some the + sign indicates a *direction* and the − sign indicates the opposite direction. Because of this, such a number, a natural number with a + or − sign placed in front of it, is called a *directed number*. They are also referred to as signed numbers. Those with the + sign are called *positive numbers;* those with the − sign are called *negative numbers.* Because later we shall introduce other kinds of numbers (of course, you have heard of fractions), we shall refer to a natural number with a + sign as a *positive integer* or positive whole number; a natural number with a − sign is a *negative integer* or negative whole number.

Probably you have noticed that we did not speak of placing a + or − sign in front of zero. Zero is the number which separates the positive numbers from the negative numbers. It has no sign. This seems quite reasonable if we realize that it is, as in the case of our figure, the label of the point from which we measure; if we are at $+3$, we are 3 feet to the right of zero; at -4 we are 4 feet to the left of zero. When we are *at* zero we are neither to the right nor to the left of zero.

Zero is the number of dollars from which we measure our assets and debts; any number of dollars greater than zero is an asset, any number of dollars less than zero is a debt. When we have neither assets nor debts we have zero dollars. Zero is the label of the altitude (sea level) from which we measure other altitudes. When we are at zero altitude we are at sea level. It is the label of the parallel of latitude (the equator) from which we measure latitude north or south. When we are at zero latitude we are on the equator. It is the meridian (through Greenwich, England) from which we measure longitude east or west. When we are at zero longitude we are on the prime meridian.

8.2 The Set of Integers

The system of numbers we now have, the set of *integers*, may be written:

$$\cdots, -5, -4, -3, -2, -1, 0, +1, +2, +3, +4, +5, \cdots$$

where the three dots at either end mean "and so on." -2 is read "negative 2," $+5$ is read "positive 5." (Some people, including us, get into the habit of reading these "minus 2" and "plus 5." We do not consider this good.)

When we had the system

$$0, 1, 2, 3, 4, \cdots$$

the numbers were arranged, from left to right, in order of size, with the smallest first. If a and b were two numbers of the system and b was to the right of a, then $a < b$ (or $b > a$) and 0 was the smallest. We cannot yet say this about our new system of numbers since in the first place we do not know what is meant by the statement $+5 > -2$, or even what is meant by $+9 > +3$ (though we probably have a good notion of the meaning of these statements). Secondly there is no "first number." The three dots on the left mean that before each number there is another number and therefore there cannot be a "first number" (just as in the old system and the new system there is no "last number").

We have several times referred to these new symbols as "numbers." There may be a question in your minds as to what right we have to do even this, that is, call such things as $+2$, -7, $+5$ "numbers." This is our game that we are constructing, and we may introduce new equipment and new rules if we think they improve the game. When baseball was first invented the catcher did not wear a mask. If he wanted to live to a ripe old age he stood some 20 feet behind the batter. Though we do not know whether the introduction of signed numbers into our mathematics will increase our longevity, we do know that it will increase the usefulness of our mathematics.

Again we should not like to introduce new equipment and new rules in such a way as entirely to change the game. We want to keep as much of our old equipment and old rules as is possible. We shall try to keep *all* our old equipment and *all* our old rules.

We attain a large part of this objective by immediately deciding that each of the numbers of our new system,

$$\cdots, -5, -4, -3, -2, -1, 0, +1, +2, +3, +4, +5, \cdots,$$

starting with 0 and going to the right, is to be identical with the corresponding number of our old system:

$$\begin{aligned} 0 &= 0 \\ +1 &= 1 \\ +2 &= 2 \\ +3 &= 3 \\ &\vdots \end{aligned}$$

In other words, the positive integers are simply new labels and new names for the natural numbers; zero remains zero.

When the numbers under discussion imply some property besides size, such as direction, then 3 shall be considered $+3$. We tell whether we are using natural numbers or positive integers by the context. The two numbers, the natural number 3 and the signed number $+3$, follow exactly the same rules. For a short while, simply for emphasis, we shall continue to use the + sign in front of positive numbers. Later we shall drop the + sign and use 3 to stand for either the natural number 3 or the positive number +3.

8.3 Addition and Subtraction of Integers

EXAMPLES 8.3.1

1. If upon a mountain with altitude +14,495 (that is, feet above sea level) a radio tower 300 feet high is built, the altitude of the top of the radio tower is $(+14{,}495) + (+300) = +14{,}795$. If a library has 14,495 books and acquires 300 more, it then has $14{,}495 + 300 = 14{,}795$ books.

2. If one mountain has an altitude of +14,495 and another has an altitude of +8673, the difference in their altitudes is +5822. That is, $(+14{,}495) - (+8673) = +5822$. But also $14{,}495 - 8673 = 5822$. If one library has 14,495 books and another has 8673 books, the first has 5822 more books than the second.

These two examples illustrate how positive integers follow the same rules for addition and subtraction as do the natural numbers. But we want to define addition and subtraction for any signed numbers. We shall give some examples and then determine what will be a good definition, a definition which will be a generalization of addition and subtraction of natural numbers.

3. If in a mountain with altitude +5295 an elevator shaft is sunk straight down from the top for a distance of 900 feet, the bottom of the shaft will have an altitude of +4395,

$$(+5295) + (-900) = +4395.$$

4. If a well is sunk in Death Valley so that its bottom is 700 feet below the floor of the valley, then, since the altitude of Death Valley is −280, the altitude of the bottom of the well will be −980,

$$(-280) + (-700) = -980.$$

5. The difference in altitude between the top of Mt. Whitney (altitude +14,495) and the floor of Death Valley (altitude −280) is +14,775,

$$(+14{,}495) - (-280) = +14{,}775.$$

You must go *up* 14,775 feet in going from Death Valley to the top of Mt. Whitney.

6. The difference in altitude between the floor of Death Valley and the top of Mt. Whitney is $-14{,}775$,

$$(-280) - (+14{,}495) = -14{,}775.$$

You must go *down* 14,775 feet in going from the top of Mt. Whitney to Death Valley.

7. If a 500-foot tower is built in Death Valley, the altitude of its top will be $+220$,

$$(-280) + (+500) = +220.$$

The top will be 220 feet *above* sea level. If a 90-foot building is built in Death Valley, the altitude of its top will be

$$(-280) + (+90) = -190.$$

Its top will be 190 feet *below* sea level.

8. The Dead Sea has an altitude of -1290. The difference between the altitude of Death Valley and that of the Dead Sea is $+1010$. Death Valley is 1010 feet *higher* than the Dead Sea,

$$(-280) - (-1290) = +1010.$$

The difference between the altitudes of the Dead Sea and Death Valley is

$$(-1290) - (-280) = -1010.$$

The Dead Sea is 1010 feet *lower* than Death Valley.

8.4 Addition of Integers on the Number Scale

The definition of the sum of two signed (directed) numbers can be understood best if we consider our figure. When we have a sum such as $(+3) + (+2)$ we start at $+3$ *and then move* 2 units to the right. The first number of the sum is the starting point, the $+$ sign for addition may be interpreted as "and then move," and the second number indicates the direction of motion and how far. In this example we arrive at 5 units to the right and so our sum is $+5$.

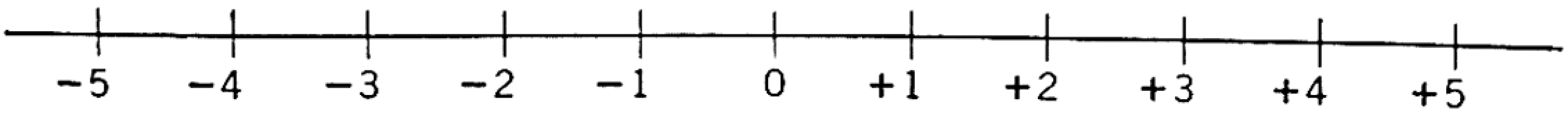


FIG. 8.4.1.

EXAMPLES 8.4.2

1. $(+1) + (+7)$ means to start at $+1$ and then move 7 units to the right, arriving at $+8$.

2. $(+7) + (+1)$ means to start at $+7$ and then move 1 unit to the right.

3. $(+5) + (-3)$ means to start at $+5$ and then move 3 units to the *left*, arriving at $+2$.

4. $(-3) + (+5)$ means to start at -3 and then move 5 units to the right.

5. $(+4) + (-7)$ means to start at $+4$ and then move 7 units to the left, arriving at -3.

6. $(-7) + (+4)$ means to start at -7 and then move 4 units to the right.

7. $(+5) + (-5)$ means to start at $+5$ and then move 5 units to the left.

8. $(-5) + (+5)$ means to start at (-5) and then move 5 units to the right.

9. $(-3) + 0$ means to start at -3 and then *not to move.*

10. $0 + (-3)$ means to start at 0 and then move 3 units to the left.

EXERCISE 8.4.3

Read and then calculate:

1. $(+7) + (+10)$.
2. $(+95) + (+105)$.
3. $(+10) + (-7)$.
4. $(-7) + (10)$.
5. $(+15) + (-17)$.
6. $(-17) + (+15)$.
7. $(-3) + (-20)$.
8. $(-28) + 0$.
9. $0 + (-28)$.
10. $[(+5) + (+5)] + (+5)$.
11. $[(+3) + (+2)] + (+7)$.
12. $(+3) + [(+2) + (+7)]$.
13. $[(+5) + (-2)] + (+9)$.
14. $(+5) + [(-2) + (+9)]$.
15. $[(-7) + (-8)] + (+15)$.
16. $(-7) + [(-8) + (+15)]$.
17. $[(-11) + (-3)] + (-9)$.
18. $(-11) + [(-3) + (-9)]$.
19. $(+7) + \{(-5) + [(-8) + (-2)]\}$.
20. $\{(+7) + [(-5) + (-8)]\} + (-2)$.

Evaluate for the indicated values of the letters:

21. $x + y;\ x = +3,\ y = -4$.
22. $[x + y] + z;\ x = -2,\ y = +3,\ z = -5$.
23. $[x + y] + z;\ x = +2,\ y = +3,\ z = -5$.

24. $x + [y + z]; x = +2, y = +3, z = -5.$
25. $x + \{[y + (x + z)] + y\}; x = +4, y = -3, z = -8.$
26. $[(x + y) + (x + z)] + (y + z); x = -5, y = +2, z = +5.$
27. $[(x + y) + z] + [x + (y + z)]; x = -4, y = +3, z = -3.$
28. $\{[(x + y) + z] + (x + y)\} + z; x = -4, y = +3, z = -3.$

8.5 Absolute Value; Addition of Integers

Before we give a definition of addition of signed numbers it will be convenient first to give a definition of what is called the "absolute value" or the "numerical value" of an integer.

8.5.1 Definition: The *absolute value* or *numerical value* of a signed integer is the natural number obtained when the sign is removed; the absolute value of zero is zero.

The absolute value (or numerical value) of a number is indicated by placing the number between vertical bars:

$$|+5| = 5,$$
$$|-5| = 5,$$
$$|0| = 0.$$

We see that the set of absolute values of the integers is the set of numbers 0, 1, 2, 3, 4, 5, $\cdots$.

Exercise 8.5.2

Read:

1. $|+3| = 3.$
2. $|-3| = 3.$
3. $|+5| + |+3| = 8.$
4. $|+5| + |-3| = 8.$
5. $|-5| + |+3| = 8.$
6. $|-5| + |-3| = 8.$
7. $|+8| - |+3| = 5.$
8. $|-8| - |+3| = 5.$
9. $|+8| - |-3| = 5.$
10. $|-8| - |-3| = 5.$
11. $|(+5) + (+3)| = 8.$
12. $|(+5) + (-3)| = 2.$

Calculate:

13. $|+2| + |-2|.$
14. $|-2| + |-2|.$
15. $(|+3| + |-3|) + |+3|.$
16. $(|-3| + |-3|) - |-3|.$
17. $|-7| - (|-7| - |-4|).$
18. $(|+8| + |-8|) - |+10|.$
19. $(|+8| + |-8|) + |-10|.$
20. $|(+8) + (-8)| + |-10|.$
21. $|+8| + (|-8| + |-10|).$
22. $(|-17| + |-17|) -$
$(|+5| + |-5|).$

8.5.3 Definition: If a and b are natural numbers, then:

$$
\begin{aligned}
(+a) + (+b) &= +(a + b),\\
(+a) + (-b) &= +(a - b), \quad \text{if } a > b\\
(+a) + (-b) &= -(b - a), \quad \text{if } a < b\\
(+a) + (-b) &= 0, \quad \text{if } a = b\\
(-a) + (-b) &= -(a + b),\\
(+a) + 0 &= +a,\\
(-a) + 0 &= -a,\\
0 + 0 &= 0.
\end{aligned}
$$

This can be restated: the sum of two integers with the same sign is an integer with this sign and equal in absolute value to the sum of the absolute values of the given integers; the sum of two integers of different signs and unequal absolute values is an integer equal to the difference between the absolute values of the integer of larger absolute value and the integer of smaller absolute value, with the sign of the integer of larger absolute value; the sum of two integers of different sign but equal absolute values is zero. Zero plays the same role as before: when it is added to a (signed) integer we get that integer; when it is added to itself we get zero.

Examples 8.5.4

1. $(+3) + (+5) = +(|+3| + |+5|) = +(3 + 5) = +8$, since both numbers have + signs.
2. $(-3) + (-5) = -(|-3| + |-5|) = -(3 + 5) = -8$, since both numbers have − signs.
3. $(+5) + (-3) = +(|+5| - |-3|) = +(5 - 3) = +2$, since the number of larger absolute value has a + sign.
4. $(-5) + (+3) = -(|-5| - |+3|) = -(5 - 3) = -2$, since the number of larger absolute value has a − sign.
5. $(+2) + (-7) = -(|-7| - |+2|) = -(7 - 2) = -5$.
6. $(-53) + 0 = -53$.
7. $(+12) + (-12) = 0$.

You should note that these results are the same as those that we obtain by using the idea of starting at the first number (on

Jammu & Kashmir University Library

our figure) and then moving right or left (or not moving if we are adding zero).

We have said that these new numbers (the signed integers) are to obey all the rules which we made for the old numbers. We first note that they obey the law of closure for addition. Also they are to obey:

8.5.5 The commutative law for addition of signed numbers:

$$(+a) + (+b) = (+b) + (+a),$$

$$(+a) + (-b) = (-b) + (+a),$$

$$(-a) + (-b) = (-b) + (-a),$$

$$(+a) + 0 = 0 + (+a),$$

$$(-a) + 0 = 0 + (-a).$$

8.5.6 The associative law for addition of signed numbers:

$$(+a) + [(+b) + (+c)] = [(+a) + (+b)] + (+c),$$

$$(+a) + [(+b) + (-c)] = [(+a) + (+b)] + (-c),$$

$$(+a) + [(-b) + (-c)] = [(+a) + (-b)] + (-c),$$

$$(-a) + [(-b) + (-c)] = [(-a) + (-b)] + (-c),$$

$$(+a) + [(+b) + 0] = [(+a) + (+b)] + 0, \text{ etc.}$$

EXERCISE 8.5.7

1 through 20. Do 1 through 20 of Exercise 8.4.3, using the *definition* (8.5.3) of addition of signed numbers.

Calculate:

21. $(+17) + [(+5) + (-22)]$.
22. $[(+17) + (+5)] + (-22)$.
23. $(+53) + [(+93) + (-16)]$.
24. $[(+53) + (+93)] + (-16)$.
25. $(+32) + [(63) + (-105)]$.
26. $(+29) + \{(+41) + [(+53) + (-41)]\}$.
27. $(+29) + \{(+53) + [(+41) + (-41)]\}$.
28. $(-32) + \{[(+75) + (-15)] + (-20)\}$.
29. $[(-32) + (+75)] + [(-15) + (-20)]$.

30. $\{[(-32) + (-15)] + (-20)\} + (+75)$.
31. $\{[(+53) + (-21)] + [(-32) + (+15)]\} + [(-63) + (-11)]$.
32. $[(+53) + (+15)] + \{[(-21) + (-32)] + [(-63) + (-11)]\}$.
33. $\{(+21) + [(-15) + (+59)]\} + \{[(+64) + (-19)] + (-22)\}$.
34. $\{(+21) + [(+59) + (64)]\} + \{[(-15) + (-19)] + (-22)\}$.

By comparing Problems 21 and 22 and by comparing Problems 23 and 24 we see applications of the associative law for addition of signed numbers.

By comparing Problems 31 and 32 and by comparing Problems 33 and 34 we see that the rule for omitting parentheses (page 9) and changing order in an addition problem also applies to the addition of signed numbers. We further see a good trick for finding the sum of signed numbers:

8.5.8 *In any addition problem, the sum of any number of signed numbers is a signed number, we may add them in any order we please, and we may group them in any order we please. Also we may find such a sum by adding all the positive numbers and adding all the negative numbers and then adding these two sums.*

Because of this we may omit in an addition problem, if we wish, all symbols of grouping; we may likewise in a sum of signed numbers insert symbols of grouping in any way that makes sense, and we may rearrange the order in any way we please.

EXERCISE 8.5.7 (continued)

Calculate:

35. $(+3) + (+5) + (-7) + (+2) + (-3)$.
36. $(+19) + (+21) + (-19) + (+5)$.
37. $(+23) + (-29) + (-63) + (+15) + (-79)$.
38. $(-42) + (-16) + (+71) + (-15) + (-79)$.
39. $(+533) + (-267) + (-721) + (-698) + (+673)$.
40. $(-333) + (+485) + (+977) + (-267) + (+862) + (-421)$.

41. $(+5288) + (-5764) + (-1234) + (-9275) + (+1492) + (+1776) + (-4832)$.

42. $(+7633) + (+4321) + (-7633) + (-1288) + (-4321) + (+1288)$.

8.6 Subtraction of Integers

We have discussed subtraction of natural numbers, and in the course of our discussion it was discovered that the natural numbers are not closed under subtraction. As far as natural numbers are concerned, $2 - 5$ is meaningless. Even when we placed zero in our system, the resulting system was not closed under subtraction. This slight generalization gave a meaning to $a - a$ only.

Also in those cases in which we have solved equations, such as Problems 20 through 27 of Exercise 6.4.2 and Problems 34 through 61 of Exercise 7.1.7, we were careful to give problems whose solutions were either natural numbers or zero. For instance, if we had the equation $x + 4 = 7$, then by definition of subtraction (of natural numbers)

$$x = 7 - 4 = 3 \text{ *}$$

or, if we had $x + 5 = 5$, then

$$x = 5 - 5 = 0.$$

But an equation such as $x + 9 = 3$ would have no solution in natural numbers or zero, since, if we try to use our definition of subtraction, we obtain $x = 3 - 9$ *which is meaningless in natural numbers.*

Whatever our definition of subtraction of signed numbers may be, it should be such that it will not contradict the definition (Definition 6.2.1) which we have for subtraction of natural numbers. This is important since we said that the positive integers and the natural numbers follow the same rules; in fact we shall write $+5$ and 5 interchangeably.

8.6.1 DEFINITION: If a and b are signed numbers, then the difference between a and b, in that order, means a signed number x such that $b + x = a$. The difference is written $a - b$, so that $x = a - b$. To subtract b from a means to find $a - b$; $a - b$ may be read "a minus b"; a is called the *minuend*, and b is called the *subtrahend.*

---

* We might also say that we solve the equation $x + 4 = 7$ by subtracting 4 from the expressions on each side of the = sign. For then we should have $(x + 4) - 4 = 7 - 4$ or $x + (4 - 4) = 3$ or $x = 3$.

If we compare this with Definition 6.2.1, we see that the only change is that we have used the words "signed number" instead of "natural number." If our signed numbers are positive (and therefore follow the same rules as the natural numbers), we can replace the words "signed numbers" by "natural numbers," and then we should have Definition 6.2.1 again.

Also, since we are using the set of integers (positive, negative, and zero), we could use the word "integer" instead of "signed number."

EXAMPLES 8.6.2

1. $(+5) - (+3)$ means a signed number x such that $(+3) + x = +5$. Of course, $x = +2$ since $(+3) + (+2) = +5$. That is, $(+5) - (+3) = +2$. Using the notation without the $+$ sign, we should have $5 - 3$ means a number x such that $3 + x = 5$. Then $x = 2$. Here each number may be considered either a natural number or a signed number (positive).

2. $(+7) - (+11)$ means a signed number x such that $(+11) + x = +7$ or $11 + x = 7$. Here $x = -4$ since $(+11) + (-4) = +7$. That is, $(+7) - (+11) = -4$ or $7 - 11 = -4$.

3. $(-3) - (+4)$ means a signed number x such that $(+4) + x = -3$ or $4 + x = -3$. Here $x = -7$ since $(+4) + (-7) = -3$.

4. $(-9) - (-3)$ means a signed number x such that $(-3) + x = -9$. Here $x = -6$ since $(-3) + (-6) = -9$.

5. $(-8) - (-15)$ means a signed number x such that $(-15) + x = -8$. Here $x = +7 = 7$ since $(-15) + 7 = -8$.

6. $(+11) - (-5)$ means a signed number x such that $(-5) + x = +11$ or $(-5) + x = 11$. Here $x = +16 = 16$ since $(-5) + 16 = 11$.

7. $(-15) - 0$ means a signed number x such that $0 + x = -15$. Here $x = -15$ or $(-15) - 0 = -15$.

8. $0 - (-15)$ means a signed number x such that $(-15) + x = 0$. Here $x = +15$ or $0 - (-15) = +15$.

8.6.3 *The integers (positive, negative, and zero) are closed under the operation of subtraction.*

EXERCISE 8.6.4

1. Tell exactly what the last statement above means.

Calculate:

2. $(+11) - (+8)$.
3. $11 - 8$. (Recall that $11 = +11$.)

4. $(+11) + (-8)$.
5. $(+7) - (+16)$.
6. $(+7) + (-16)$.
7. $(-7) + (-16)$.
8. $(+9) - (+9)$.
9. $9 - 9$.
10. $(+9) + (-9)$.
11. $(+9) - (-9)$.
12. $(+9) + (+9)$.
13. $(+17) - (-11)$.
14. $(+17) + (+11)$.
15. $(+5) - (-38)$.
16. $(+5) + (+38)$.
17. $(+32) - 0$.
18. $(-51) - 0$.
19. $0 - (-32)$.
20. $0 - (+51)$.
21. $0 - 0$.

Evaluate for the indicated values of the letters:

22. $x + y; x = 12, y = -15$.
23. $x + y; x = -12, y = 15$.
24. $x - y; x = 12, y = -15$.
25. $x - y; x = -12, y = 15$.

26. $(x + y) + z; x = 5, y = -7, z = 3$.
27. $(x + y) - z; x = 5, y = -7, z = -3$.
28. $(x - y) + z; x = 5, y = -7, z = -3$.
29. $(x - y) + z; x = -5, y = -7, z = -3$.
30. $x + (y + z); x = 5, y = -7, z = 3$.
31. $x + (y + z); x = 5, y = -7, z = -3$.
32. $x - (y + z); x = 5, y = -7, z = -3$.
33. $x - (y - z); x = -5, y = 7, z = 3$.
34. $x - (y + z); x = -5, y = 7, z = 3$.

8.7 A Rule for Subtraction

If we compare certain of these problems, such as Problems 2 and 4, 5 and 6, 8 and 10, we see what appears to be a rule:

8.7.1 *To subtract a signed number from a signed number we need only change the sign of the subtrahend and proceed to add.*

That is, subtracting a positive number gives the same result as adding the corresponding negative number, and subtracting a negative number gives the same result as adding the corresponding positive number.

That this is a correct rule follows from the definition of subtraction. For instance, if we consider the case of subtracting a positive number from a positive number, by Definition 8.6.1, if a and b are natural numbers, $(+a) - (+b) = x$, a signed number such that $+a = (+b) + x$. But, by using the associative and commutative laws, we see that:

$$\begin{aligned}(+b) + [(+a) + (-b)] &= [(+b) + (-b)] + (+a)\\ &= 0 + (+a)\\ &= +a,\end{aligned}$$

so that the number x which must be added to $+b$ to get $+a$ is $(+a) + (-b)$, or

$$(+a) - (+b) = (+a) + (-b),$$

which is what we asserted.

EXERCISE 8.7.2

1. Show that this rule is correct for subtracting a positive from a negative number. That is, show that

$$(-a) - (+b) = (-a) + (-b).$$

2. Show that

$$(+a) - (-b) = (+a) + (+b).$$

3. Show that

$$(-a) - (-b) = (-a) + (+b).$$

Calculate:

4. $(+73) - (-57)$.
5. $(+73) - (+57)$.
6. $(-95) - (+67)$.
7. $(-95) - (-67)$.

8. $[(+23) - (-64)] - (-58)$.
9. $[(-54) - (+73)] - (+92)$.
10. $[(-49) - (+51)] - (+100)$.
11. $(+35) - [(-42) - (+42)]$.
12. $(+98) - [(-77) - (-77)]$.

13. $83 - (63 - 75)$.
14. $99 - (165 - 101)$.
15. $111 - [73 - (-192)]$.
16. $53 - (92 + 70)$.
17. $53 - (92 - 70)$.
18. $(75 - 99) - (84 - 37)$.
19. $(75 - 99) + (84 - 37)$.
20. $79 - [32 + (53 - 39)]$.
21. $79 - [32 - (53 - 39)]$.
22. $79 - [32 + (53 + 39)]$.

23. $95 - \{54 - [72 - (172 - 5)]\}$.
24. $95 + \{(-54) + [(-72) + (172 - 5)]\}$.
25. $95 - \{(-54) - [(-72) - (172 - 5)]\}$.
26. $95 + \{54 + [(-72) + (-172 + 5)]\}$.
27. $95 + \{54 + [72 + (172 - 5)]\}$.
28. $95 + \{54 - [(-72) + \overline{(-172) + 5}]\}$.

Since subtracting a positive number gives the same result as adding the corresponding negative number, when we have an expression such as $5 - 7$ we may consider it either as meaning $(+5) - (+7)$ or as meaning $(+5) + (-7)$. The minus sign may be considered either a sign of subtraction or as part of the symbol for negative 7, which we then add.

Similarly, since subtracting a negative number gives the same result as adding the corresponding positive number, when we have an expression such as $9 - (-4)$, we may consider this as either $(+9) - (-4)$ or as $(+9) + (+4)$ or simply $9 + 4$.

By introducing the idea of the *negative of a number* we can state our rule more simply.

8.7.3 DEFINITION: The negative of a signed number x shall be the number which when added to x gives the sum 0. The negative of x is indicated by $-x$.

Symbolically the negative of $+5$ is $-(+5)$. But the number which when added to $+5$ gives the sum 0 is the number -5. We see therefore that

$$-(+5) = -5.$$

This is read, "the negative of positive 5 is negative 5."

Symbolically the negative of -7 is $-(-7)$. But the number which when added to -7 gives the sum 0 is the number $+7$. Therefore

$$-(-7) = +7,$$

or the negative of negative 7 is positive 7.

Symbolically the negative of 0 is -0. But the number which added to 0 gives the sum 0 is 0. Therefore,

$$-0 = 0.$$

In all cases we see that, if a is a natural number, then:

$$\begin{aligned} -(+a) &= -a, \\ -(-a) &= +a, \\ -0 &= 0. \end{aligned}$$

8.7.4 Our rule now becomes: *To subtract a signed number from a signed number, add the negative of the subtrahend to the minuend.*

Or: *If a and b are signed numbers, then $a - b = a + (-b)$.* Problems 4 through 28 of Exercise 8.7.2 illustrate this rule.

Referring again to Figure 8.4.1, in order to find $+3 - (+5)$ using the definition of subtraction, we must find the number x such that $+3 = +5 + x$. We must answer the question: How many units must we move and in what direction from the point $+5$ to get to the point $+3$? The answer is 2 units to the left so that $+3 - (+5) = -2$.

Looking at it from the point of view of adding the negative of the subtrahend, we start at $+3$ and then move 5 units to the left, arriving at -2.

To find $(+5) - (-3)$ we must find the number x such that $+5 = (-3) + x$. How many units and in what direction must we move from -3 to arrive at the point $+5$? [10]

Changing it to the problem of adding the negative of the subtrahend, we have $(+5) + (+3)$. Interpret this on our figure.

The following remark has been made: "Whereas negative numbers were invented to make the operation of subtraction possible without any exception, after we have negative numbers we no longer need to use the operation of subtraction." Explain why this is a correct remark.[11]

8.8 Removing Parentheses in Addition and Subtraction Problems

Frequently, as in Problems 11 through 28 of Exercise 8.7.2, we have to subtract a group of numbers. For example,

$$-4 - [(+5) + (-8)].$$

This problem asks us to subtract the sum of $+5$ and -8 from -4; that is, to subtract -3 from -4 or to find $-4-(-3)=-1$. But we could get the same result by finding the negative of each number within the symbol of grouping and adding,

$$-4+(-5)+(+8)=(-9)+(+8)=-1.$$

This is an example of a general rule similar to the rule for removing parentheses in a problem involving addition only:

8.8.1 In any problem involving additions and subtractions only, a grouping symbol preceded by a + sign may be removed without any other change; a grouping symbol preceded by a − sign may be removed provided that all signs within the group be changed, + signs being replaced by − signs and − signs by + signs.

EXAMPLES 8.8.2

1. $15-(7+3)=15-7-3$. This, of course, may be interpreted as $+15+(-7)+(-3)$.
2. $32+(17-6)=32+17-6$.
3. $63-(82-15+7)=63-82+15-7$.
4. $-(71+25-63)+(17-2+11)=-71-25+63+17-2+11$.
5. $a+b-(c-d+e)=a+b-c+d-e$.
6. $32-[53-(62-7)]=32-[53-62+7]=32-53+62-7$.

EXERCISE 8.8.3

Calculate in two ways (first by combining numbers within grouping symbols and second by removing all symbols of grouping and then combining the numbers):

1. $8-(3-5)$.
2. $15-(9-2)$.
3. $99-(165-101)$.
4. $(75-99)-(84-37)$.
5. $(75-99)+(84-37)$.
6. $79-[32+(53-39)]$.
7. $79-[32-(53-39)]$.
8. $87-[54+(16-25)-(32+17)]$.
9. $-27-(54-27)$.
10. $-(63-72+54)+(-63+72-54)$.

11. $73 - \{[62 - 74 + (81 - 9)] - [51 + (67 - 71)]\}$.
12. $-\{42 - [-38 + (-25 + 37) - (52 - 5)]\} + 1$.

Solve the equations:

13. $x - (5 - 7) = (8 - 2)$.
14. $a - (9 - 6) = (6 - 9)$.
15. $p - (11 - 2) = -(11 - 2)$.
16. $x - (17 - 4) = (19 - 2)$.
17. $y + 3 = 5$. (Note that we may look upon this from the point of view of the definition of subtraction or as $y - (-3) = 5$.)
18. $r + (17 - 2) = (8 - 2)$.
19. $x + (19 - 26) = (35 - 28)$.
20. $q + [11 - (5 - 17)] = 18 - [17 - (19 - 7)]$.

Evaluate for $x = 3$, $y = 2$, $z = -4$:

21. $x - (y + z)$.
22. $x + (y - z)$.
23. $x + (-y + z)$.
24. $x - (-y + z)$.
25. $x + (-y - z)$.
26. $x - (-y - z)$.
27. $x - [y + (x + z)]$.
28. $x - [y - (x + z)]$.

Remove all symbols of grouping and combine like terms:

29. $a + (b - a)$.
30. $x - (y - x)$.
31. $3p - (p + q)$.
32. $2x + (x - y)$.
33. $3a - (2a - b)$.
34. $2s - (t + 4s)$.
35. $8 + (5x - 3)$.
36. $17 - (11 - 2t)$.
37. $(5 - 3x) + (5x - 3)$.
38. $(2a + b) - (a - b)$.
39. $(p + 2q) - (p - 2q)$.
40. $a + [b + (a - b)]$.
41. $p + [q - (p - q)]$.
42. $x - [y + (x - y)]$.
43. $r - [s - (r + s)]$.
44. $3t - [t - (t - 4)]$.
45. $10 - [x + (2 - x)]$.
46. $3b - [b + (b - 2)]$.
47. $13 + [t - (7 + 2)]$.
48. $(r - s) + [2s - (s - r)]$.

49. $(2x - 4) - [3x + (4 - x)]$.
50. $(3a - 2b) - [2a - (2b - a)]$.
51. $(5x^4 + 4x^3 - 3x^2 + 2x - 1) - (2x^4 - 3x^3 - 2x^2 + 4x + 2)$.
52. $(3x^3 - x + 7) - (3 - 2x + 4x^2 + 3x^3)$.
53. $4x^3 - [x^3 - 2x^2 - (x^2 + 2x - 3)]$.
54. $7x^4 + [3x^4 - 2x^3 - (4x^4 - 3x^3 + 2x^2 - 1)]$.
55. $x^5 + 3x^4 - [2x^3 - (x^2 - 4x + 6)]$.
56. $2x - \{x^2 + [3x^3 - (2x^2 - 4x + 1) + 3] - 5x\}$.

8.9 "Greater Than" and "Less Than" for Integers

When we first introduced signed numbers we raised the question of the meaning of $+5 > -2$ and of $+9 > +3$. We want our meanings of "greater than" and "less than" for signed numbers to be generalizations of their meanings for natural numbers.

8.9.1 DEFINITION: If a and b are signed numbers, then $a > b$ if $a - b$ is positive, $a < b$ if $a - b$ is negative, $a = b$ if $a - b$ is zero.

Thus:

$$+9 > +3 \text{ since } (+9) - (+3) = +6,$$

$$+5 > -2 \text{ since } (+5) - (-2) = +7,$$

$$-2 > -7 \text{ since } (-2) - (-7) = +5,$$

$$-7 < -2 \text{ since } (-7) - (-2) = -5.$$

Another way of saying this, by referring to our number scale (Figure 8.4.1), is: a is greater than b, if a lies to the right of b; a is less than b, if a lies to the left of b.

EXERCISE 8.9.2

Determine which quantity is greater:

1. $5 - (3 - 2)$ or $19 - (32 - 16)$.
2. $11 + 52 - (17 - 3)$ or $7^2 - 2$.
3. $19 - 37$ or $5 - 21$.
4. $(18 - 59) - (97 - 30)$ or $57 - (68 - 44 + 60)$.
5. $21 - (52 - 3)$ or $31 - [63 - (72 - 59)]$.
6. $(9 - 3^2) - (7 - 5^2)$ or $(9 - 3)^2 - (7 - 5)^2$.
7. $(9 - 3)^2 - (7^2 - 5)$ or $(9^2 - 3) - (7 - 5)^2$.
8. $(18 - 7)(21 - 11)$ or $(18 - 15)^3(17 - 15)^2$.

Read each of the following:

9. $-3 < x < -1$.
10. $-3 \leqq x < -1$.
11. $-3 < x \leqq -1$.
12. $-3 \leqq x \leqq -1$.
13. $-5 < x \leqq 0$.
14. $-5 \leqq x < 0$.

Express each of the following in symbols:

15. x is less than 3 but not less than -2.
16. x is greater than -2 but not greater than 3.
17. x is not less than -2 and not greater than 3.
18. x is greater than -4 and less than or equal to 0.
19. x is less than or equal to -2 and not less than -8.
20. x is not greater than -2 and greater than or equal to -8.

9

The Integers; Multiplication

9.1 Laws of Signs

We have seen that, for natural numbers, multiplication is merely repeated addition; $(+3)(+4) = 3 \cdot 4 = 4 + 4 + 4 = +12 = 12$; $a \cdot b = b + b + b + \cdots + b$, where there are a b's in the sum. (Here the symbols can be thought of as either natural numbers or as positive integers.)

We can carry this idea over to signed numbers in the case of such products as

$$3 \cdot (-4) = (-4) + (-4) + (-4) = -12.$$

We want all our rules to carry over from natural numbers to signed numbers; in particular we want the commutative law for multiplication to hold for these numbers. In order that this be so we must have

$$(-4) \cdot 3 = 3 \cdot (-4) = -12.$$

9.1.1 Definition: *The product of a negative integer and a positive integer, in either order, is a negative integer equal in absolute value to the product of the absolute values of the factors.*

Note that this is a *definition* of the product of two signed numbers with opposite signs.

When the two signed numbers involved in multiplication are both negative we have a more difficult decision to make. We certainly want the absolute value of the product to equal the product of the absolute values. For $(-3)(-4)$ we want the absolute value of the product to be 12. But what sign should we attach to this product? Should the product be $+12$ or -12? We decide on $+12$ for various reasons.

We want the distributive law to hold for signed numbers, that is $a(b + c) = ab + ac$. If we set $a = -1$, $b = +1$, $c = -1$, we obtain

$$(-1)[(+1) + (-1)] = (-1)(+1) + (-1)(-1).$$

Now the first term on the right is -1 by our last previous definition. If $(-1)(-1) = -1$, then we have on the right $(-1) + (-1) = -2$. But the expression on the left is $(-1) \cdot 0 = 0$. And, following the rule $(-1)(-1) = -1$, we are led to the contradiction $0 = -2$. For this reason we decide to let $(-1)(-1) = +1$.

Another reason for deciding to let the product of two negatives be a positive is an application of what is known as the "principle of permanence of formal laws." * This principle states that we "employ rules under circumstances more general than are warranted by the special cases under which the rules were derived and have validity." An example of the application of this principle again leads to the conclusion that it would be convenient to have the product of two negatives a positive.

Suppose that a, b, c, d are natural numbers and $a > b$ and $c > d$. Then $a - b$ and $c - d$ are natural numbers.

We wish to examine the product

$$(a - b) \cdot (c - d).$$

We have never before considered such a product and therefore decide to examine it by considering a geometrical figure. We draw a rectangle of length a and width c, and then draw the lines at distances b and d from one end and one side, respectively, as shown in the figure. The unshaded rectangle has length $a - b$ and width $c - d$, and therefore area $(a - b)(c - d)$. Its area is the product in which we are interested. But this area certainly equals the area of the whole figure less the shaded areas. From the area $a \cdot c$ we must take away $a \cdot d$ and $b \cdot c$; in doing this we have taken away $b \cdot d$ twice and therefore must put it back. We then have

$$(a - b)(c - d) = a \cdot c - a \cdot d - b \cdot c + b \cdot d.$$

---

* Felix Klein: *Elementary Mathematics from an Advanced Standpoint*, Vol. I, The Macmillan Co., New York. Translated and copyrighted by E. R. Hedrick and C. A. Noble (1932). See also George Peacock: *A Treatise on Algebra*, reprint of the 1845 edition, Vol. II, page 18, Scripta Mathematica, 1940.

Now this result was obtained under the assumption that a, b, c, d are natural numbers and therefore can be used to represent the lengths and widths of rectangles, and also under the assumption that $a > b$ and $c > d$. There is sense in speaking of $a = 10$, $b = 3$, $c = 7$, $d = 2$. Such dimensions have meaning on our figure. There would be no sense in referring to our figure if, say, $a = -11$ or if $a = 5$ and $b = 7$. We cannot have a rectangle of length -11, nor is $5 - 7$ a length. Therefore we cannot say that we have shown that our expression for $(a - b)(c - d)$ is correct for such values. But, if we assume the *permanence* of the result, or, in other words, if we assume that the result shall hold for all values of a, b, c, d, then

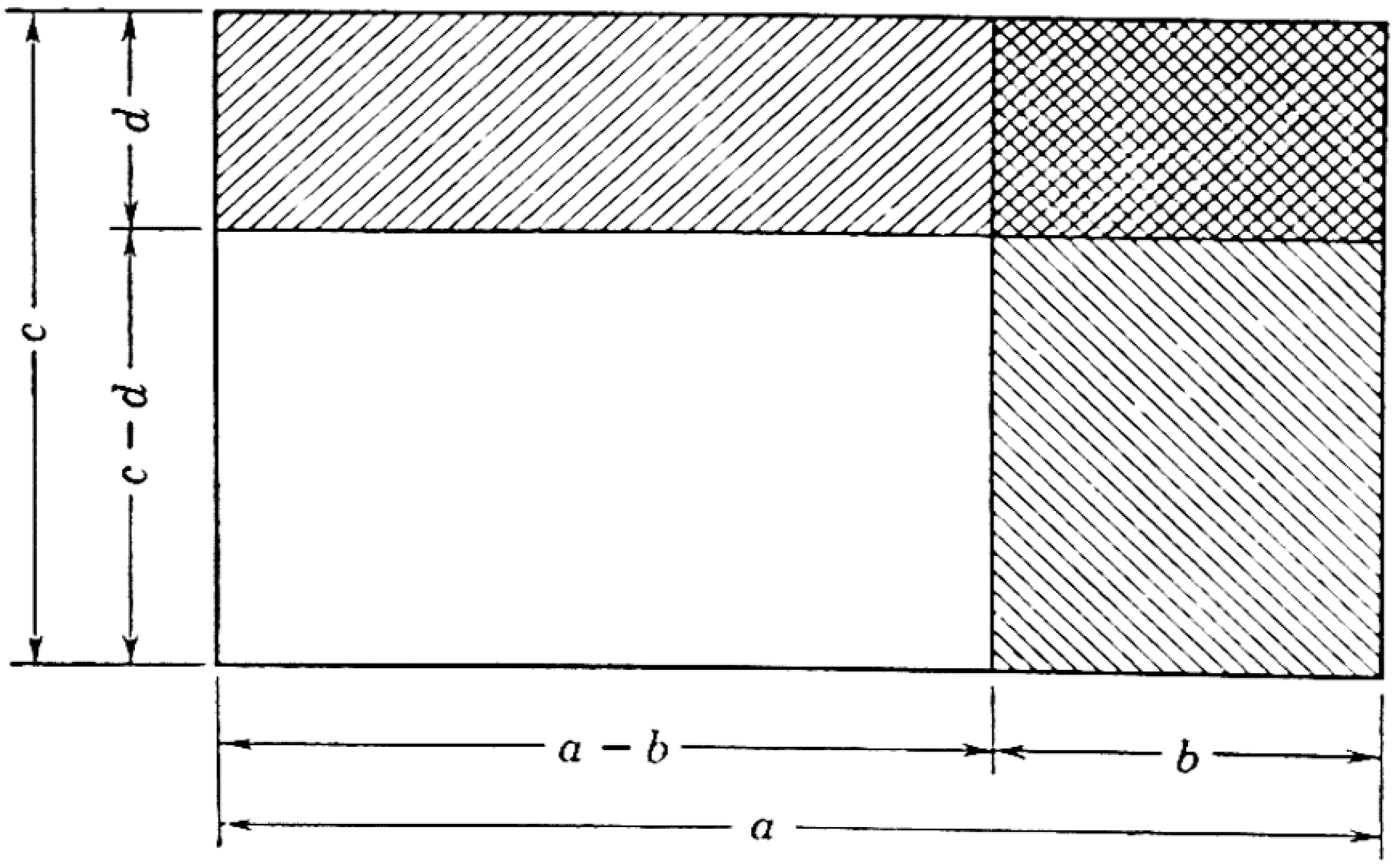


FIG. 9.1.2.

$$(a - b)(c - d) = ac - ad - bc + bd,$$

no matter what numbers we substitute for a, b, c, d.

Let us then set $a = 0$ and $c = 0$. Then we have

$$(0 - b)(0 - d) = 0 \cdot 0 - 0 \cdot d - b \cdot 0 + bd$$

or

$$(-b)(-d) = +bd.$$

We must emphasize that we have not *proved* this rule. We have merely given two reasons why it seems to be a sensible rule. Indeed, as we have stated before, such rules cannot be proved, any more than one can prove that "three strikes is out," or that a "full house beats a flush." We state the *definition:*

9.1.3 DEFINITION: *The product of two negative integers is a positive integer equal in absolute value to the product of the absolute values of the factors.*

We state the rules concerning multiplication of integers as the

9.1.4 LAW OF SIGNS FOR MULTIPLICATION OF INTEGERS: The product of two integers of like signs is a positive integer; the product of two integers of unlike signs is a negative integer. The product of any integer and zero is zero.

EXAMPLES 9.1.5

1. $(+2)(+3) = +6$ or $2 \cdot 3 = 6$.
2. $(+2)(-3) = -6$ or $2 \cdot (-3) = -6$.
3. $(-2)(+3) = -6$ or $-2 \cdot 3 = -6$.
4. $(-2)(-3) = +6$.
5. $(-4) \cdot 0 = 0$.
6. $(+4) \cdot 0 = 0$.

EXERCISE 9.1.6

Calculate:

1. $(-5)(+3)$.
2. $(+5)(-3)$.
3. $(-7)(-2)$.
4. $(-8)(-3) + (+8)(+3)$.
5. $(+7)(-4) - (-3)(+3)$.
6. $3[(-2) + 4]$ (two ways).
7. $-4(2 - 5)$ (two ways).
8. $(-1)^2$.
9. $(-1)^3$.
10. $(-1)^{12}$.
11. $(-1)^{17}$.
12. $(-2)^3 + (-3)^2$.
13. $-3[9 - (-3)^2]$.
14. $4[5^3 + (-5)^3]$.
15. $(-3)^2 \cdot (-2)^3$.
16. $5^2 \cdot (-1)^5 + 5^2$.
17. $-2^3 + (-2)^3$.
18. $-2^4 + (-2)^4$.
19. $[(5 - 7)^2 - (7 - 5)^2] \cdot (-2383)$.
20. $[(-3)^3]^2$.
21. $-[(7 - 2)^2 - (2 \cdot 5)^2]$.
22. $2 - 2 \cdot 3 \cdot 5^2$.
23. $2 - 2(-3 \cdot 5)^2$.
24. $2 - (-2)(-3)(-5)^2$.

Read and then multiply out:

25. $x(x - y)$.
26. $a(2a - 3b)$.
27. $p^2(p^2 - pq + q^2)$.
28. $-r(r - s)$.
29. $(-x)^2(x^2 + xy + y^2)$.
30. $-a^3(a^2 - ab + b^2)$.
31. $(-a)^3(a^2 - ab + b^2)$.
32. $-p^4(p^2 - pq + q^2)$.
33. $(-p)^4(p^2 - pq + q^2)$.
34. $(-r)^2(-s)^3(s^2 - r^2)$.
35. $(-t^2)^3(t^2 - 2t + 1)$.
36. $(-t^3)^2(t^2 - 2t + 1)$.
37. $[(-t)^2]^3(t^2 - 2t + 1)$.
38. $-x^2y^3(x^3 - 3x^2y + 3xy^2 - y^3)$.
39. $(-x)^2y^3(x^3 - 3x^2y + 3xy^2 - y^3)$.
40. $x^2(-y)^3(x^3 - 3x^2y + 3xy^2 - y^3)$.

Multiply out and combine like terms:

41. $a + b - 2(3a + 4b) + 3(2a - b)$.
42. $3p - (4p + 2q) - 3[p - 2(p - q)]$.
43. $-s(t + u) + 2u(s - t) - 3t(u + s)$.
44. $3p^2(p - q) + 2pq(p - 2q) - q^2(2p - q)$.
45. $3 - 2(a + 2b) + 4(2a - 3)$.
46. $7(p - 3q) - 5(p - 2) - 4(q + 3)$.
47. $3x^2 - 2[x + 2x(x - 3)]$.
48. $r^3[s^2 - r(s - t)] - 2r^2[r^2s + r(rt - s^2)]$.
49. $2 - 3\{a - k + 2[3 - 3a + 2k - 3(a - k)] - a\}$.
50. $(a - b)(a - c)$.
51. $(a - b)(a - b) - (a + b)(a + b)$.
52. $(x - y)(x + y)$.
53. $p^2(q - r) - q^2(r - p) + r^2(p - q)$.
54. $2x^4[x - y^2(x - z^2) + 3z^2(x - y^2)] - 5x^5$.
55. $-2p[3p^3 - 2p^2 + 2p - 4 + p(p^2 - 3p + 1)]$.
56. $r\{4 - st[(r - s^2 + 2t) + t(r + 3s - 2s^2)]\} - 5rst$.
57. $a^5(a^2b + ab^2) - a^4(a^3b + ab^3) + a^3(a^4b + ab^4) - a^2(a^5b + ab^5) + a(a^6b + ab^6)$.
58. $x^4y^4(x^2y - xy^2) - 2x^3y^3(x^3y^2 - x^2y^3) + 3x^2y^2(x^4y^3 - x^3y^4) - 4xy(x^5y^4 - x^4y^5) + 5(x^6y^5 - x^5y^6)$.
59. $(3x^2 - 2x + 1) - 3(x^2 + 2x - 1)$.
60. $-2(x^3 - 4x^2 + 5) + 5(x^2 + 2x + 1)$.
61. $5[x^2 + x(x + 3)] - x(x^2 + 1)$.
62. $-7x[x + x(x - 1)] - 6x^2[x^2 + x^2(x - 1)]$.
63. $-3x^3\{-x^2 + x[1 - x(2x^2 - x - 1)]\}$.
64. $4x^2\{x^3 - x[2x^2 - 3x(x - 1)] + x^2[3x - x(x + 1)] - 3\}$.

Take out the highest common factor:

65. $x^2y - xy^2$.
66. $p^3r^2 - p^2r^3$.
67. $ab^2c^3 - a^3b^2c$.
68. $u^2v^3w^4 - u^4v^2w^3 + u^3v^4w^2$.
69. $3x^2y^2 - 5x^3y^3 + 4x^4y^3z$.
70. $-2a^3b^2 + 4ab^3 - 8a^2b^4$.
71. $9pq^2r^2 - 27p^3q^2r + 3^4pq^3r^3$.
72. $3^2st^2 - 5^2s^2t$.
73. $25ab^8c^3 - 75a^2b^5 - 125b^3c^3$.
74. $6x^2(y - z)^3 - 3x^3(y - z)^2$.
75. $(u - v)^2(u^2 + v^2) - (u - v)(u^2 + v^2)^2$.
76. $x(x - y) - y(x - y)$.
77. $x^2 - 2xy + y^2$.
78. $a^2 - 6a + 9$.

Solve the following equations and verify that your solutions are correct (note that the difficulty mentioned in the footnote to 7.1.6 has now been eliminated):

79. $(t - 1)(t - 3) = 0$.
80. $(p - 3)(p - 6) = 0$.
81. $(x + 4)(x - 1) = 0$.
82. $(x + 3)(x + 7) = 0$.
83. $(a - 2)(a + 2) = 0$.
84. $(x - 1)(x + 1)(x + 3) = 0$.
85. $(x + 1)(x + 2)(x + 3)(x + 4) = 0$.
86. $(r - 3)(r + 3)(r - 4)(r + 4) = 0$.
87. $(s - 2)(s + 2)(t - 1)(t + 1) = 0$.
88. $(x - 1)^2(x + 1)^2 = 0$.

10

The Integers; Division

10.1 Division Defined

We have considered the three operations, addition, multiplication, and subtraction. In order to have a system of numbers which is closed under all three operations we enlarged our original system of (natural) numbers by inventing the number zero and the signed integers (positive and negative). Another reason for introducing the signed integers was so that equations of the type $x + a = b$, where a and b are natural numbers, would have solutions regardless of the relative sizes of a and b.

It was pointed out that subtraction is the inverse of addition, that is, subtraction "undoes" addition. It was also pointed out that division undoes multiplication and is therefore said to be the inverse of multiplication. In doing the problem $72 \div 9$ we are answering the question, 9 times what number is 72. This illustrates the process of division. To divide a number by another number means to find the number which when multiplied by the second gives us the first number. To divide a by b means to find the number x such that $bx = a$.

10.1.1 DEFINITION: If a and b are numbers, the *quotient* of a and b, in that order, means a number x such that $b \cdot x = a$. The quotient is written $a \div b$, so that $x = a \div b$. To divide a by b means to find the quotient $a \div b$; a is called the *dividend* and b is called the *divisor*.

To divide correctly and quickly we must know the multiplication tables!

EXAMPLES 10.1.2

1. $6 \div 3$ means a number x such that $3 \cdot x = 6$. Of course, $x = 2$ in this case; that is, $6 \div 3 = 2$.

2. $54 \div 9$ means a number x such that $9 \cdot x = 54$. Here $x = 6$.

3. $(-12) \div 3$ means a number x such that $3 \cdot x = -12$. Here $x = -4$; $(-12) \div 3 = -4$ since $3 \cdot (-4) = -12$.

4. $12 \div (-3)$ means a number x such that $-3 \cdot x = 12$. Here $x = -4$; $12 \div (-3) = -4$ since $(-3)(-4) = 12$.

5. $(-15) \div (-5)$ means a number x such that $(-5) \cdot x = -15$. Here $x = +3$ or 3; $(-15) \div (-5) = +3 = 3$ since $(-5) \cdot 3 = -15$.

6. $0 \div 7$ means a number x such that $7 \cdot x = 0$. Here $x = 0$; $0 \div 7 = 0$ since $7 \cdot 0 = 0$.

7. $0 \div (-9)$ means a number x such that $(-9) \cdot x = 0$. Here $x = 0$; $0 \div (-9) = 0$ since $(-9) \cdot 0 = 0$.

8. $5 \div 0$ means a number x such that $0 \cdot x = 5$. Is there such a number x?

9. $-7 \div 0$ means a number x such that $0 \cdot x = -7$. Is there such a number x?

10. $0 \div 0$ means a number x such that $0 \cdot x = 0$. How many numbers x can you find to do this?

11. $3 \div 5$ means a number x such that $5 \cdot x = 3$. Is there a number x of our system (the integers) which will do this?

12. $-7 \div 2$ means a number x such that $2 \cdot x = -7$. Is there an integer x which will do this?

If you know the multiplication tables, you know that the answer to the question in Examples 11 and 12 is "no" in each case. There is no number x of the system

$$\cdots, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, \cdots$$

such that $5 \cdot x = 3$ nor is there a number x of the system such that $2 \cdot x = -7$. We see immediately that the set of integers is *not* closed under division. (State exactly what this means.)

10.1.3 DEFINITION: In those cases in which the quotient of a and b is a number of our system, that is, when $a \div b = x$, an integer, we say that a is a *multiple* of b or that b is a *divisor* or *factor* of a. (Read Definition 2.4.1 again.)

EXAMPLES 10.1.4

1. 63 is a multiple of 9, and 9 is a divisor or factor of 63. Why? [12]

2. 10 is a multiple of 10, and 10 is a divisor or factor of 10. Why? [13]

3. 6 is a divisor or factor of −48, and −48 is a multiple of 6. Why? [14]

4. −33 is a multiple of −3, and −3 is a divisor or factor of −33. Why? [15]

5. −11 is a divisor or factor of −33, and −33 is a multiple of −11. Why? [16]

6. Since $5 \cdot 6 = 6 \cdot 5 = 30$ we may say that *both* 5 and 6 are divisors or factors of 30. Are there any other divisors or factors of 30? [17]

7. 60 is a multiple of 1, 4, 5, and 10. We say that 60 is a *common* multiple of 1, 4, 5, and 10. Is 60 the *smallest* (or least) common multiple of 1, 4, 5, and 10?

Since division is the inverse of multiplication we should expect that a law of signs for division is obtainable by consideration of the law of signs for multiplication of signed numbers.

If a and b are natural numbers, then $(+a) \div (+b)$ means a (signed) number x such that $+a = (+b) \cdot x$. Since $+b$ times this unknown number x is a positive number, x must also be positive. The quotient of a positive number and a positive number is a positive number.

By considering $(+a) \div (-b)$, $(-a) \div (-b)$, and $(-a) \div (+b)$ in similar ways we reach the conclusion:

10.1.5 *The quotient of two numbers of like sign is a positive number; the quotient of two numbers of unlike signs is a negative number; the quotient of zero and any number except zero is zero.*

EXERCISE 10.1.6

Read each of the following; show that each is correct:

1. $72 \div 8 = 9$.

2. $-132 \div 11 = -12$.

3. $99 \div (-9) = -11$.

4. $-125 \div (-5) = 25$.

5. $(x^2 - y^2) \div (x - y) = x + y.$
6. $(x^2 - x - 2) \div (x + 1) = x - 2.$
7. $(x^2 + 2xy + y^2) \div (x + y) = x + y.$
8. $(x^2 - 2xy + y^2) \div (x - y) = x - y.$
9. $(77 \div 7)\cdot 7 = 77.$
10. $(x^2 - x - 2) \div (x - 2) = x + 1.$
11. $x^5 \div x^2 = x^3.$
12. $a^9 \div a^3 = a^6.$
13. $r^6 \div r^6 = 1.$
14. $(x + 1)^3 \div (x + 1) = (x + 1)^2.$
15. $(p^7q^6) \div (p^2q^4) = p^5q^2.$
16. $[(r - s)^5(r + s)^5] \div (r^2 - s^2)^5 = 1.$
17. $9 + (18 \div 3) = 15.$
18. $(9 + 18) \div 3 = 9.$
19. Show that, if $x = (+a) \div (-b)$, where a and b are natural numbers, then x is negative.
20. Show that, if $x = (-a) \div (-b)$, where a and b are natural numbers, then x is positive.
21. Show that, if $x = (-a) \div (+b)$, where a and b are natural numbers, then x is negative.

10.2 Order of Operations

We have agreed that, in a problem involving addition, subtraction, and multiplication, the multiplication precedes the addition and subtraction unless parentheses indicate otherwise. Thus

$$3 - 2\cdot 4 + 5(6 - 2) + 11 = 3 - 8 + 5\cdot 4 + 11$$
$$= 3 - 8 + 20 + 11 = 26.$$

10.2.1 We make the further agreement that *in a problem involving addition, subtraction, multiplication, and division, the multiplications and divisions are to precede the additions and subtractions unless parentheses indicate otherwise.*

Problem 17 above may be written without parentheses

$$9 + 18 \div 3 = 15.$$

But remember our admonition that it is better to have more parentheses than necessary than to have too few so that the problem may be misunderstood.

In some cases involving both multiplication and division there may be doubt as to the meaning of the symbols unless parentheses are used. For instance, if we have

$$8 \div 2 \cdot 4,$$

since there is no agreement as to which, multiplication or division, is to take precedence, this will have the value 1 if the multiplication is done first and the value 16 if the division is done first.* Again, if we have

$$2 \cdot 8 \div 4,$$

this will have the value 4 no matter which, the multiplication or the division, is done first. In such cases we use parentheses to make our meaning clear:

$$(8 \div 2) \cdot 4 = 16;$$

$$8 \div (2 \cdot 4) = 1.$$

When in doubt use parentheses.

Exercise 10.2.2

Calculate:

1. $12 + 18 \div 6$.

2. $(12 + 18) \div 6$.

3. $7 + 6 \cdot 9 + 15 \div 3$.

4. $7 + \overline{[6(9 + 15)} \div 3]$.

5. $7 + 6[(9 + 15) \div 3]$.

6. $7 + (6 \div 3)(9 + 15)$.

7. $19 - [(10 + 17) \div 3]9$.

8. $19 - (10 + 17) \div (3 \cdot 9)$.

9. $4[18 \div 3 + 3] - 4[18 \div (3 + 3)]$.

10. $16 \div [9 \div (7 - 4) + 1]$.

11. $16 \div [9 \div (4 - 7) + 1]$.

12. $16 \div [-9 \div (4 - 7) + 1]$.

Evaluate for the given values:

13. $x \div y + 3;\ x = 18,\ y = 3$.

14. $x \div y - 3;\ x = 18,\ y = -3$.

---

* "Read from left to right" is a rule of *English*, not of algebra. There are languages which are read from right to left and also languages which are read from top to bottom. We know of no language which is read from bottom to top, though there is no particular reason why such a language cannot be devised.

15. $x - y \div x;\ x = -4,\ y = -12.$
16. $(x - y) \div x;\ x = -4,\ y = -12.$
17. $x \div y - x;\ x = -24,\ y = 3.$
18. $x \div (y - x);\ x = -24,\ y = -20.$
19. $2x + y \div (x - y);\ x = 18,\ y = 12.$
20. $3x - x \div 2y + x;\ x = 12,\ y = 6.$
21. $(2x \div y)\cdot x;\ x = 4,\ y = -2.$
22. $2x \div (y\cdot x);\ x = 4,\ y = -2.$
23. $xy - x \div y;\ x = 12,\ y = -3.$
24. $x(x + y) + x \div (x + y);\ x = 18,\ y = -12.$
25. $x[y(x + y)] - x \div [y(x - 6y)];\ x = 24,\ y = 2.$
26. $(-x + y) \div (3x + 4y);\ x = 5,\ y = -2.$
27. $(5x - 5y) \div (y - x);\ x = -10,\ y = 4.$
28. $(9x^2 - y^2) \div (y + 3x);\ x = -3,\ y = 4.$

Write in mathematical symbols:

29. The quantity x squared minus y squared divided by the quantity x minus y equals x plus y.

30. The quantity x minus y, squared, divided by the quantity x minus y equals x minus y.

31. Bracket, the quantity a minus b to the fifth power, times the quantity a plus b to the fifth power, close bracket, divided by the fifth power of the quantity a squared minus b squared equals one.

32. The quantity x plus 3 divided by 4, plus the quantity x plus 1 divided by the quantity x minus 1 equals the quantity x squared plus $6x$ plus 1 divided by four times parentheses x minus 1.

Assuming the correctness of the following equalities, make *four* statements, two using the words *multiple of* and two using the words *divisor of* (or *factor of*):

33. $7\cdot 3 = 21.$
34. $-9\cdot 5 = -45.$
35. $(-11)(-6) = 66.$
36. $36 \div 3 = 12.$
37. $-42 \div 3 = -14.$
38. $39 \div (-13) = -3.$

39. $(x - y)(x + y) = x^2 - y^2.$
40. $(x - 2)(x + 1) = x^2 - x - 2.$
41. $(x - y)(x^2 + xy + y^2) = x^3 - y^3.$
42. $(x^3 + y^3) \div (x + y) = x^2 - xy + y^2.$
43. $(x^2 - 5x + 6) \div (x - 2) = x - 3.$

44. $x^5 \cdot x^{11} = x^{16}$.
45. $(xy)^2 \cdot (x^3y^3) = (xy)^5$.
46. $x^7 \div x^2 = x^5$.
47. $(x^3y^5) \div (x^2y^4) = xy$.

Solve:

48. $5x = 15$.
49. $7x = -28$.
50. $-2x = 10$.
51. $-9x = -45$.

52. $3x + 8 = 23$. (Recall the definition of subtraction.)

53. $2t + 15 = 1$.
54. $5z + 6 = 66$.
55. $13a + 65 = 0$.
56. $12t + 72 = 0$.
57. $8x + 51 = 67$.
58. $3x + 47 = 5$.
59. $5(x + 3) = 25$.
60. $20 - 7y = 6$.
61. $92 - 11y = 70$.
62. $3(y + 1) + 4 = 13$.
63. $2(x + 1) + 17 = 13$.
64. $2(x + 2) + 3(x + 3) = 23$.
65. $5(x - 3) + 2(x + 9) = 17$.
66. $4(3x + 1) + 3(2x - 1) = 55$.
67. $7(2x - 3) + 5(3x - 2) = 27$.
68. $3x + 2(x + 1) = -13$.
69. $6 + 2y = 5y$.

Set up the equations and solve them:

70. The sum of a certain number and twice that number is 18.

71. Four times 3 more than a certain number is 20.

72. $43 is divided among Tom, Dick, and Harry. Tom receives twice as much as Dick, and Harry receives $3 more than Tom. How much does each receive?

73. Joe is twice as old as John, and the sum of their ages is 42.

74. A cord 59 inches long is cut into three pieces of which the first is twice as long as the second and the third is four inches longer than the first. How long is each piece?

75. A man has $80 in $1, $5, and $10 bills, the same number of each. How many does he have of each?

76. What three consecutive natural numbers have a sum of 18?

77. A father earns twice as much each day as his son. The father works 8 days and the son works 5 days. They receive together payment of $84 for their work. How much does each receive per day? What is the total amount each receives?

78. The length of a rectangle is twice its width. What are its length and width if its perimeter is 66 feet?

79. The perimeter of a rectangle is three times its length. What is its length if its width is 12 feet? What is its perimeter?

80. The number of square feet in the area of a rectangle is the same as the number of feet in its perimeter. What is its width if its length is 6 feet? What is its area? its perimeter?

81. At the school election a total of 454 votes were cast for the two candidates for president, Albert and Paul. Paul won by 58 votes. How many votes did each get?

82. A ball is thrown downward from a balloon with an initial speed of 40 feet per second. Each second that the ball falls its speed increases by 32 feet per second. In how many seconds will its speed reach 168 feet per second?

83. In a child's bank there is a total of $3.82 in pennies, nickels, dimes, and quarters. There are six more dimes than quarters, two more nickels than six times the number of quarters, and twice as many pennies as nickels. How many coins of each kind are in the bank?

10.3 The Role of Zero in Division

Examples 6 and 7 of Examples 10.1.2 indicate that $0 \div 7$ and $0 \div (-9)$ are each zero. In general, if b is a number different from zero then $0 \div b = 0$. Since $0 \div b$ means a number x such that $b \cdot x = 0$, x must be zero if b is not zero (7.1.5).

10.3.1 *Zero divided by any number not zero is zero* or, *if* $b \neq 0$, *then* $0 \div b = 0$.

When we first introduced the number zero we stated that it had all the properties of other numbers except one property. This exception is indicated by Examples 8, 9, and 10 of Examples 10.1.2.

Recall the definition of a quotient. If a and b are numbers, the quotient of a and b, in that order, means a number x such that $b \cdot x = a$.

Suppose $a \neq 0$. Then $a \div 0$ would mean a number x such that $0 \cdot x = a$. But we know that 0 times any number is 0, and therefore the left side of the equality is zero while the right side is not zero. This, of course, is a contradiction.

If we now consider $0 \div 0$, if it means anything, it means a number x such that $0 \cdot x = 0$. But x can then be any number whatsoever, and the symbol $0 \div 0$ is useless. In order that a symbol be useful it must represent something definite; it must convey an idea to our minds. Since $0 \div 0$ does not represent any definite number no one can tell what is meant if it is used. Therefore, we never use it.

10.3.2 *Division by zero is never permitted* or, *if a is a number, then $a \div 0$ is not a number.*

Examples 10.3.3

1. $7 \div 0$ is meaningless.
2. $-19 \div 0$ is meaningless.
3. $3 \div (x - 1)$ is meaningless if $x = 1$.
4. $9 \div (x^2 - 1)$ is meaningless if $x = 1$ or $x = -1$.
5. $0 \div (x - 1) = 0$ if $x \neq 1$; if $x = 1$, it is meaningless.
6. $0 \div (x^2 - 4) = 0$ if $x \neq 2$ and if $x \neq -2$. If $x = 2$ or $x = -2$, it is meaningless.
7. $[(x - 1)(x + 1)] \div (x - 1) = x + 1$ if $x \neq 1$; if $x = 1$, it is meaningless.
8. $(x - 2) \div (x - 2) = 1$ if $x \neq 2$; if $x = 2$, it is meaningless.

Exercise 10.3.4

For what integer values of x is each of the following meaningless:

1. $5 \div (x - 3)$.
2. $-3 \div (x + 4)$.
3. $0 \div (x - 7)$.
4. $0 \div (x + 9)$.
5. $2 \div x$.
6. $(x + 1) \div (x - 1)$.
7. $(x + 2) \div (x^2 - 4)$.
8. $3 \div (x^2 + 1)$.
9. $0 \div (x^2 + 2)$.
10. $0 \div x^2$.
11. $(x^2 + 1) \div (x^2 - 1)$.
12. $(x^2 - 1) \div (x^2 - 1)$.
13. $(x + 2) \div (x + 2)^2$.
14. $x \div (1 - x)$.
15. $(x - 1) \div (1 - x)$.
16. $3 \div (9 - x^2)$.
17. $3 \div (9 - x)^2$.
18. $(9 - x^2) \div (9 - x)^2$.
19. $(9 - x)^2 \div (9 - x^2)$.
20. $2 \div (x^2 - x - 2)$.

11

Fractions

11.1 Fish, Farms, and Fractions

Long years ago, in prehistoric times, two friends, Ugh and Onestone, went on a fishing expedition. They agreed, before they started, to divide the catch equally between them. They threw each fish that they caught into a common pile. At the end of the day they divided the fish by taking a fish and placing it to one side, then taking another fish and placing it on the other side, and repeating this process until they had two piles. But when they came to the end they had one fish left over. They pondered the problem for quite a while. Finally Onestone took his ax and split the extra fish into two pieces, threw one piece on his friend's pile, and took the other piece for himself. They returned to their caves still good friends.

This happened several times and with different things. Each time that there was an extra fish or rabbit or apple Onestone would solve the problem by cutting the extra into two pieces. Later they were joined by another friend on an expedition, and when there was an extra fish Onestone solved the problem by cutting it into three pieces. A really brain-racking problem arose when there were two extra fish. Onestone solved this problem also, but only after much thought. He cut each extra fish into three pieces, and each friend was given two of these pieces.

Now we doubt that this is more than a "fish story." But it illustrates how the notion of fractions might have arisen from mundane, practical problems.

Later, after man moved out of caves and began to build houses and lay out land for farming, it became increasingly necessary to answer the question, How long? A stick of a certain length may have been used as a unit of measurement. The number of times that it could be laid end to end along the edge of a piece of land was a measure of the length of the piece of land. Sometimes it "came out even," that is, the last time that the measuring stick was placed in position its end and the end of the piece of land coincided. More frequently this did not happen.

The measuring stick was then broken or *fractured* into equal smaller lengths, and these smaller units or *fractions* were used to try to come out even. For instance, the unit stick may have been laid down five times, but the piece of land extended beyond the stick the fifth time that it was placed in position. The extra length was too small to place the whole length of stick again. The unit stick was broken in half. The half stick did not quite come to the end of the land, and so the other half was broken in half again and one of these smaller fractional pieces was laid down. This was repeated, each time one of the fractions being broken to make a smaller fraction. Eventually the end of the land and one of the fractional pieces seemed to coincide or to come so close to coinciding that the difference seemed unimportant. The length of the piece of land may have been noted as 5 units and one-half unit and one half of one-half unit and one half of one half of one-half unit. It became the law of the tribe that after three successive fractures of the unit stick the nearest half of one half of one half was to be accepted as the length. Today we should say that the lengths were measured to the nearest eighth of a unit.

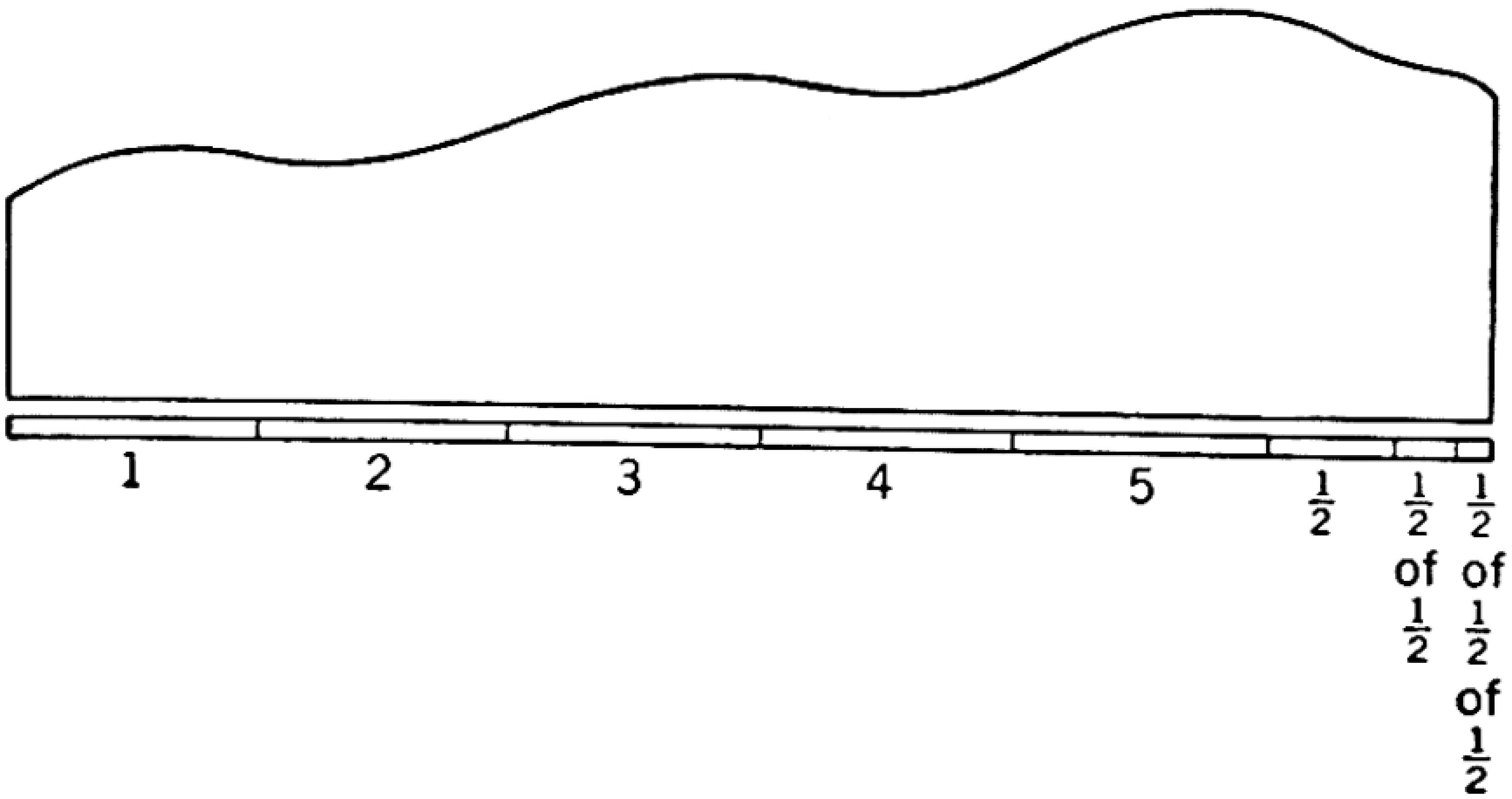

1
2
3
4
5
$\frac{1}{2}$
$\frac{1}{2}$ of $\frac{1}{2}$
$\frac{1}{2}$ of $\frac{1}{2}$ of $\frac{1}{2}$

FIG. 11.1.1.

We have pointed out that $3 \div 5$ and $-7 \div 2$ are not numbers of our system (of integers):

$$\cdots, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, \cdots;$$

and therefore $3 \div 5$ and $-7 \div 2$ are, for the moment, without meaning. The symbol $a \div b$ is, for the moment, without meaning if b is not a factor of a, that is, if there does not exist an integer x such that $a = bx$.

We shall now formally introduce new numbers which will give the symbol $a \div b$ meaning in all cases (except for $b = 0$). As before, when we brought new numbers into our game, we shall want these numbers to follow the same rules as our old numbers.

11.1.2 DEFINITION: A fraction is a symbol $\frac{a}{b}$ or a/b, where a and b are natural numbers. It is read "a over b"; a is called the *numerator*, and b is called the *denominator*.

EXAMPLES 11.1.3

1. $\frac{1}{2} = 1/2$ is a fraction.
2. $\frac{2}{3} = 2/3$ is a fraction.
3. $\frac{7}{5} = 7/5$ is a fraction.
4. $\frac{19}{2} = 19/2$ is a fraction.
5. $\frac{5}{0}$ is not a fraction since 0 is not a natural number. We shall soon see another reason why we do not consider this a fraction. Indeed this symbol will never have a meaning.
6. $\frac{0}{0}$ is not a fraction. This symbol also will never have a meaning.*

In our story of land measuring by means of a unit stick and half of half the stick, by a half we meant one of two equal parts. In general a/b shall mean a of b equal parts.

7. $1/3$ means 1 of 3 equal parts into which we have divided something, for instance, a measuring stick.
8. $2/3$ means 2 of 3 equal parts.

---

* The symbol $0/0$ is used in calculus. It is used there as a shorthand symbol for a process, the process of determining what happens to a fraction when the numerator and denominator each get close to zero but do not equal zero. A question which might arise in calculus is: What happens to the fraction $(x - 1)/(x^2 - 1)$ when x gets close to 1 but is not equal to 1? We see that when x gets close to 1 both the numerator and the denominator get close to zero but are not equal to zero. Find the values of this fraction when $x = 1.2$, 1.1, 1.01, 1.001, 0.99, 0.999. To what value does the fraction seem to get close as x gets close to 1? [18]

9. $3/3$ means 3 of 3 equal parts, and therefore it is equal to the whole of something we started with.

10. $4/3$ means 4 of 3 equal parts. This may sound silly on first reading. But we mean by it that we have divided as many as we need of something, each into 3 equal parts, and are considering 4 of those parts. For instance, we might cut each of 2 pies into 3 slices and eat 4 of the slices. We have then eaten $4/3$ of a pie (and may wish we had not).

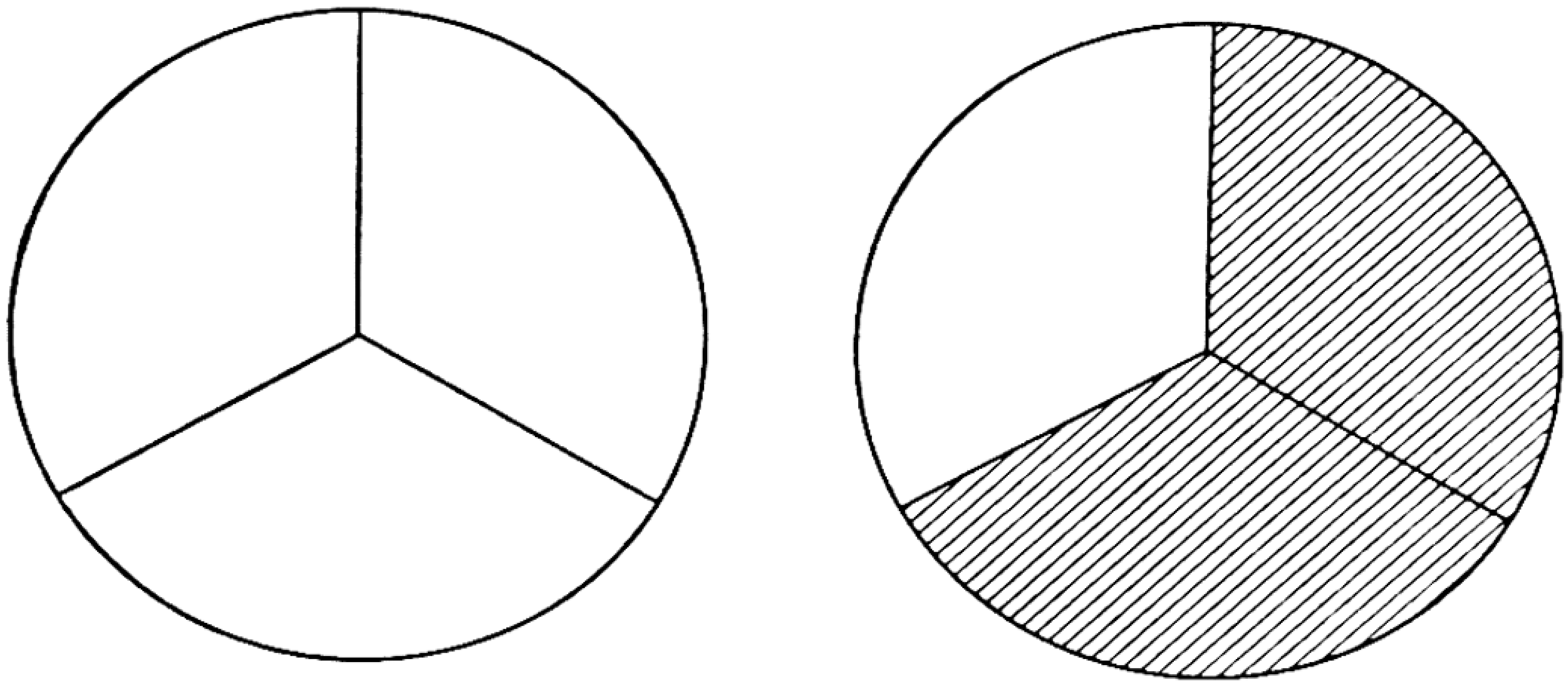


FIG. 11.1.4. The unshaded part is $\frac{4}{3}$ of a pie. How much is the shaded part?

11. $17/6$ means 17 of 6 equal parts.

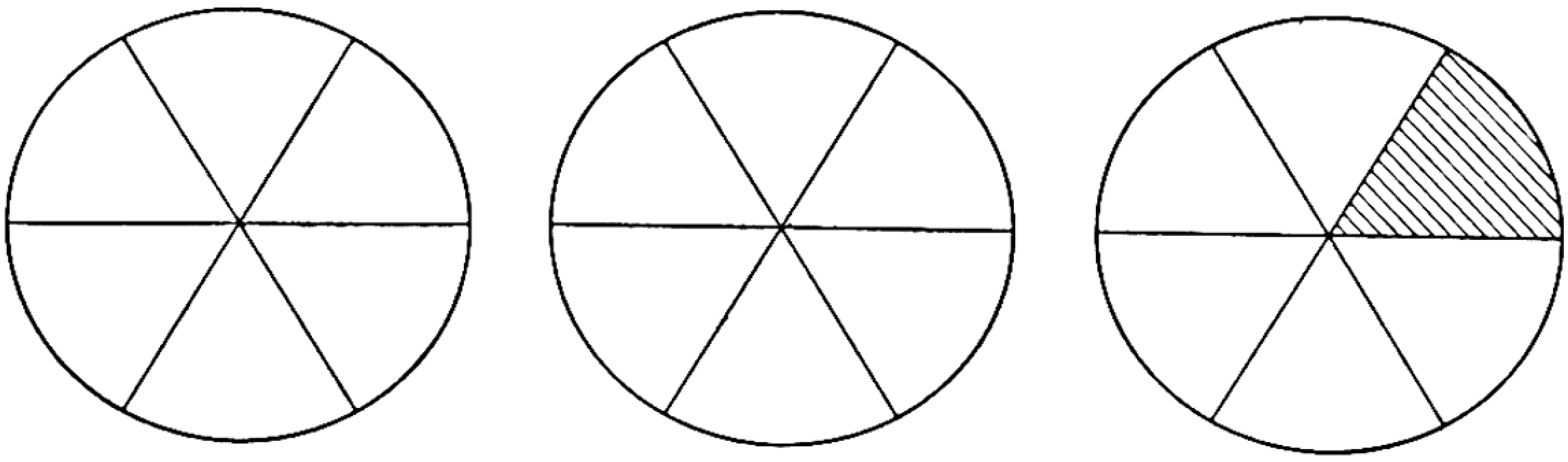
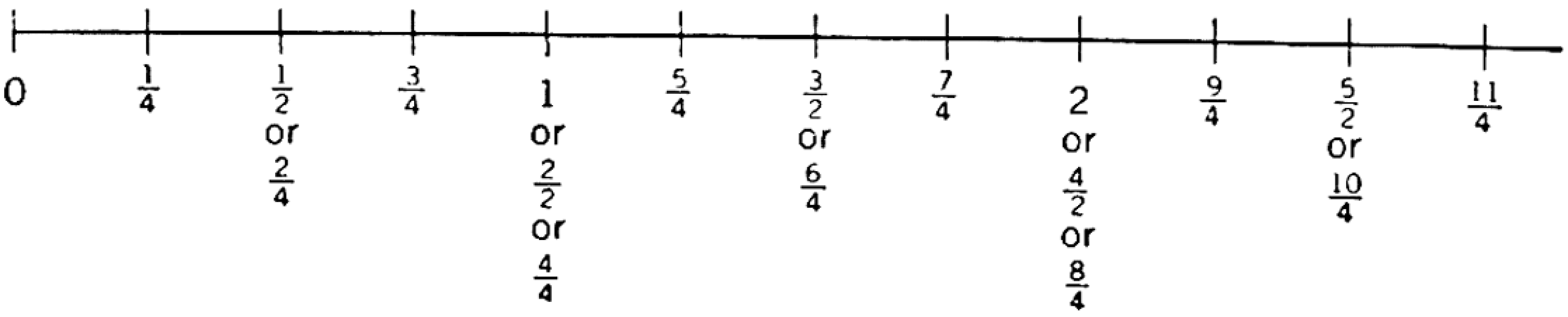


FIG. 11.1.5. The unshaded part is $\frac{17}{6}$ of a pie.

11.2 Equal Fractions

On this figure we have labeled the second "fracture" of our first measuring stick "$1/2$ or $2/4$." This certainly seems correct since if



0
$\frac{1}{4}$
$\frac{1}{2}$ or $\frac{2}{4}$
$\frac{3}{4}$
1 or $\frac{2}{2}$ or $\frac{4}{4}$
$\frac{5}{4}$
$\frac{3}{2}$ or $\frac{6}{4}$
$\frac{7}{4}$
2 or $\frac{4}{2}$ or $\frac{8}{4}$
$\frac{9}{4}$
$\frac{5}{2}$ or $\frac{10}{4}$
$\frac{11}{4}$

FIG. 11.2.1.

we have broken the stick into 4 equal parts 2 of these parts will be equal in length to the piece obtained when we broke it into only 2 equal parts, or

$$\tfrac{1}{2} = \tfrac{2}{4};$$

similarly, we can see that

$$\tfrac{3}{2} = \tfrac{6}{4}.$$

If we break our unit stick into 5 pieces and compare it with what we obtain when we break it into 10 pieces, we see that

$$\tfrac{1}{5} = \tfrac{2}{10},$$

$$\tfrac{2}{5} = \tfrac{4}{10}, \text{ etc.}$$

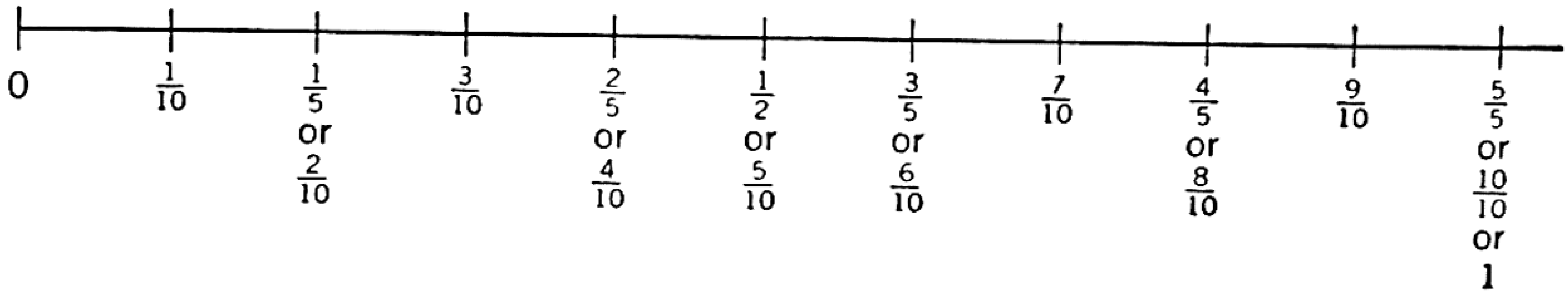


FIG. 11.2.2.

This suggests the

11.2.3 POSTULATE: If a, b, and x are natural numbers, then

$$\frac{ax}{bx} = \frac{a}{b}.$$

EXAMPLES 11.2.4

1. $\dfrac{15}{35} = \dfrac{3}{7}$ since $\dfrac{15}{35} = \dfrac{3\cdot 5}{7\cdot 5}$.
2. $\dfrac{3}{9} = \dfrac{1}{3}$ since $\dfrac{3}{9} = \dfrac{1\cdot 3}{3\cdot 3}$.
3. $\dfrac{15}{10} = \dfrac{3}{2}$ since $\dfrac{15}{10} = \dfrac{3\cdot 5}{2\cdot 5}$.
4. $\dfrac{21}{7} = \dfrac{3}{1}$ since $\dfrac{21}{7} = \dfrac{3\cdot 7}{1\cdot 7}$.
5. $\dfrac{2}{3} = \dfrac{2\cdot 7}{3\cdot 7} = \dfrac{14}{21}$.
6. $\dfrac{9}{5} = \dfrac{9\cdot 3}{5\cdot 3} = \dfrac{27}{15}$.
7. $\dfrac{2}{3} = \dfrac{2\cdot 5}{3\cdot 5} = \dfrac{10}{15}$ and $\dfrac{2}{3} = \dfrac{2\cdot 7}{3\cdot 7} = \dfrac{14}{21}$ and $\dfrac{2}{3} = \dfrac{2\cdot 8}{3\cdot 8} = \dfrac{16}{24}$.

This last example gives us several fractions, each of which is equal to $2/3$ by Postulate 11.2.3. In each case the new fraction is obtained from $2/3$ by multiplying the numerator and the denominator by the same number. Since each of these fractions equals $2/3$, it follows that we may write $10/15 = 14/21 = 16/24$. [Remember what the $=$ sign means, page 7.]

However, if we have fractions such as $10/15$ and $14/21$, it may be difficult to tell at once that they are equal. We cannot obtain $14/21$ from $10/15$ directly by using Postulate 11.2.3 since there is no natural number x such that $10x = 14$ and $15x = 21$. We can use this postulate to write two fractions, one obtained from $10/15$ and the other obtained from $14/21$, such that these two new fractions are equal to each other.

There are many such fractions obtainable. For instance, we might make use of our postulate to say

$$\frac{14}{21} = \frac{2 \cdot 7}{3 \cdot 7} = \frac{2}{3}$$

and

$$\frac{10}{15} = \frac{2 \cdot 5}{3 \cdot 5} = \frac{2}{3},$$

and, since when expressed with denominator 3 they each have numerator 2, they are equal. Also

$$\frac{14}{21} = \frac{14 \cdot 5}{21 \cdot 5} = \frac{70}{105}$$

and

$$\frac{10}{15} = \frac{10 \cdot 7}{15 \cdot 7} = \frac{70}{105},$$

and again when our two given fractions are expressed with equal denominators they also have equal numerators.

11.2.5 *Two fractions shall be equal if and only if when they are expressed with equal denominators they have equal numerators or if when they are expressed with equal numerators they have equal denominators.*

We use Postulate 11.2.3 either from right to left or from left to right. In the first case we are multiplying the numerator and the denominator by the same number; in the second case we are dividing the numerator and the denominator by the same number. This is of such great importance that we state it as a rule:

11.2.6 *The only operations we may perform on a fraction so as to obtain another fraction equal to the given fraction is either to multiply both numerator and denominator by the same number (not zero) or to divide both numerator and denominator by the same number.*

To express a given fraction with a certain different denominator we follow the steps:

1. Answer the question: What must the denominator be multiplied by or divided by to obtain the new denominator?
2. Multiply or divide the numerator by the same number.

Example: Express $5/8$ with the denominator 24.

1. The denominator must be multiplied by 3.
2. We multiply the numerator also by 3. The result is $15/24$.

Example: Express $16/12$ with denominator 3.

1. The denominator must be divided by 4.
2. We also divide the numerator by 4. The result is $4/3$.

Sometimes we must go through two steps.

Example: Express $24/15$ with denominator 20. We can neither multiply nor divide 15 by a (natural) number to obtain 20. But, if we divide 15 by 3 and then multiply the result by 4, we obtain 20. We perform the same two operations on 24. The result is $32/20$.

To express a given fraction with a certain different numerator we follow the same steps but interchange the words "numerator" and "denominator."

EXERCISE 11.2.7

Express the given fraction as an equal fraction with the indicated denominator or numerator:

1. $\frac{3}{4} = \frac{\quad}{12}$. 2. $\frac{5}{3} = \frac{\quad}{12}$. 3. $\frac{7}{12} = \frac{14}{\quad}$. 4. $\frac{11}{6} = \frac{44}{\quad}$.
5. $\frac{4}{6} = \frac{\quad}{3}$. 6. $\frac{18}{15} = \frac{\quad}{5}$. 7. $\frac{24}{18} = \frac{4}{\quad}$. 8. $\frac{35}{7} = \frac{\quad}{1}$.
9. $\frac{66}{22} = \frac{\quad}{1}$. 10. $\frac{3}{6} = \frac{\quad}{8}$. 11. $\frac{21}{14} = \frac{\quad}{10}$. 12. $\frac{15}{33} = \frac{35}{\quad}$.
13. $\frac{36}{45} = \frac{\quad}{60}$. 14. $\frac{60}{24} = \frac{70}{\quad}$.

We know that

$$\frac{24}{60} = \frac{12\cdot 2}{30\cdot 2} = \frac{12}{30} = \frac{6\cdot 2}{15\cdot 2} = \frac{6}{15} = \frac{2\cdot 3}{5\cdot 3} = \frac{2}{5}$$

and we cannot express $^{24}/_{60}$ with a denominator smaller than 5. When a fraction has been expressed with the smallest possible denominator (a natural number) we say that it has been *reduced to lowest terms.* Of course, $^{24}/_{60}$ could be reduced to lowest terms in fewer steps than we have taken:

$$\frac{24}{60} = \frac{2\cdot 12}{5\cdot 12} = \frac{2}{5};$$

12 is the *highest common factor* of 24 and 60, meaning it is a factor of both 24 and 60 and is the largest number which is a factor of both 24 and 60. (Refer to page 31.)

Reduce to lowest terms:

15. $\frac{12}{60}$.
16. $\frac{30}{10}$.
17. $\frac{40}{64}$.
18. $\frac{55}{33}$.
19. $\frac{12}{132}$.
20. $\frac{72}{108}$.
21. $\frac{160}{140}$.
22. $\frac{196}{35}$.
23. $\frac{57}{95}$.
24. $\frac{105}{70}$.
25. $\frac{63}{63}$.
26. $\frac{4620}{8008}$.

Reduce to lowest terms; evaluate the result for the given values of the letters. Check this by substituting the values of the letters in the original expression and then reducing to lowest terms.

27. $\dfrac{x^5}{x^7}$; $x = 2$.
28. $\dfrac{a^7}{a^4}$; $a = 3$.
29. $\dfrac{xy}{x^2}$; $x = 1, y = 2$.
30. $\dfrac{x^3y^2}{x^2y}$; $x = 2, y = 1$.
31. $\dfrac{x^4y^2z}{x^2y^3z^2}$; $x = 1, y = 2, z = 3$.
32. $\dfrac{2x^2y^5}{6xy^6}$; $x = 5, y = 1$.
33. $\dfrac{6a^3b^2}{8a^4b}$; $a = 2, b = 3$.
34. $\dfrac{56p^3q^3}{70p^4q^2}$; $p = 2, q = 3$.
35. $\dfrac{(2x^2)^3}{4x^5}$; $x = 3$.
36. $\dfrac{(x^3y^2)(x^2y^3)}{(x^3y^2)^2}$; $x = 1, y = 2$.
37. $\dfrac{(a^2b)^2(ab^2)^2}{(a^2b^2)^3}$; $a = 1, b = 3$.

38. $\dfrac{[(r^2s^3)^2]^3}{[(r^3s^2)^3]^2}$; $r = 2$, $s = 3$.

39. $\dfrac{18^2 \cdot 25^2}{4^2 \cdot 10^3}$; (hint: $18 = 3^2 \cdot 2$).

40. $\dfrac{(9x)^3}{(3x^2)^4}$; $x = 3$.

41. $\dfrac{x^2 + x}{x^3 + x^2}$; $x = 2$.

42. $\dfrac{7a - 7}{56a}$; $a = 1$.

43. $\dfrac{ab + ac}{b^2 + bc}$; $a = b = c = 2$.

44. $\dfrac{3xy + 6xz}{6y^2 + 12yz}$; $x = y = z = 3$.

45. $\dfrac{(x + y)^2}{(x + y)(x - y)}$; $x = 2$, $y = 1$.

46. $\dfrac{x^2 + 2xy + y^2}{x^2 - y^2}$; $x = 5$, $y = 1$.

11.3 Prime Numbers

In reducing fractions to lowest terms we have divided the numerator and denominator by the highest common factor of these two numbers. We shall now digress for a short while to determine a method of finding this important quantity.

In order to understand our method we must know something more about natural numbers. We first define a *prime number*.

11.3.1 DEFINITION: A *prime number* is a number which has exactly two positive factors, itself and one.

EXAMPLES 11.3.2

1. 2 is a prime number. It has as positive factors 2 and 1 only.

2. 3 is a prime number.

3. 17 is a prime number.

4. 4 is not a prime number. Besides 4 and 1 it also has the factor 2.

5. 6 is not a prime number. It has the four positive factors 1, 2, 3, and 6. Such numbers as 4 and 6, with *more* than two positive factors, are said to be *composite*.

6. 1 is not a prime number; it has only itself as a positive factor. It also is not composite. 1 is sometimes called a units number.

Prime numbers have held a fascination for mathematicians, both professional and amateur, throughout the ages. One of the first questions which might arise is, How many prime numbers are there? If we start writing the prime numbers in order of size

$$2, 3, 5, 7, 11, 13, 17, 19, 23, \cdots,$$

do the three dots at the right mean "and so on" in the same sense as when we put them to the right of the list of natural numbers:

$$1, 2, 3, 4, 5, \cdots?$$

Do they mean that we can continue on and on? We get the next natural number from a given one simply by adding 1 to it. But no such simple method seems obvious for obtaining the next prime number from a given one.

Another way of asking our question is, Is there a largest prime number? The answer is, No! We shall not prove this here but point out that a proof of this fact has been known for a long time. The best-known proof is attributed to Euclid, the great Greek mathematician of about 300 B.C.*

There is no known method of determining the "next prime." If we have a prime, say 1877, there is no simple way, other than trial, to find the next larger one. However, if we set the problem: list all the primes less than a certain number, there is a comparatively simple way to do this. This method is known as the Sieve of Eratosthenes.

Suppose that we want a list of all the primes less than 50. We write the numbers from 2 through 50:

	2	3	~~4~~	5	~~6~~	7	~~8~~	~~9~~	~~10~~
11	~~12~~	13	~~14~~	~~15~~	~~16~~	17	~~18~~	19	~~20~~
~~21~~	~~22~~	23	~~24~~	~~25~~	~~26~~	~~27~~	~~28~~	29	~~30~~
31	~~32~~	~~33~~	~~34~~	~~35~~	~~36~~	37	~~38~~	~~39~~	~~40~~
41	~~42~~	43	~~44~~	~~45~~	~~46~~	47	~~48~~	~~49~~	~~50~~

We leave the 2 and then proceed to cross out every second number after it. We leave the next uncrossed-out number, 3, but cross out every third number. We leave the next uncrossed-out number, 5, but cross out every fifth number, and so on. When we finish, the numbers *not* crossed out are all prime. We note that after crossing out every *seventh* number we are finished with our job in this case.

---

* See, e.g., Wright: *Theory of Numbers*, page 9, or Uspensky and Heaslet: *Elementary Number Theory*, page 85. As of February, 1952, the largest *known* prime number is $180(2^{127} - 1)^2 + 1$. See *Scientific American* for February, 1952, page 40.

In going through this process we come to numbers which have already been crossed out. These are counted, but, of course, there is no point in crossing them out twice. For instance, 12 is crossed out when we are crossing out every second number. It also is a number to be crossed out when we are crossing out every third number.

Many other interesting questions concerning prime numbers can be raised, and you might enjoy looking further into the matter.*

Now suppose that we consider the number 12 and its factors:

$$12 = 6 \cdot 2 = 3 \cdot 4 = 3 \cdot 2 \cdot 2 = 2 \cdot 3 \cdot 2 = 2 \cdot 2 \cdot 3 = 2^2 \cdot 3.$$

We see that 12 has the *prime* factors 2 and 3 and that 2 is "a factor twice." Other than for order, when 12 is written in terms of its *prime* factors there is only one way to write it. This illustrates a fundamental rule:

11.3.3 *Every composite number can be expressed as a product of prime numbers in one and only one way, except for the order in which the factors are written.*

EXAMPLES 11.3.4

1. $15 = 3 \cdot 5$ or $5 \cdot 3$ but these are the same except for order.
2. $18 = 2 \cdot 3 \cdot 3 = 3 \cdot 2 \cdot 3 = 3 \cdot 3 \cdot 2 = 2 \cdot 3^2$.
3. $16 = 2 \cdot 2 \cdot 2 \cdot 2 = 2^4$.
4. $13 = 13$.
5. $36 = 2 \cdot 2 \cdot 3 \cdot 3 = 2^2 \cdot 3^2$.
6. $60 = 2^2 \cdot 3 \cdot 5$.
7. $300 = 2^2 \cdot 3 \cdot 5^2$.
8. $360 = 2 \cdot 180 = 2^2 \cdot 90 = 2^3 \cdot 45 = 2^3 \cdot 3 \cdot 15 = 2^3 \cdot 3^2 \cdot 5$.

EXERCISE 11.3.5

1. Use the Sieve of Eratosthenes to write all the primes less than 100.
2. Write all the primes less than 150.

---

* See, e.g., Richardson: *Fundamentals of Mathematics*, page 202, The Macmillan Co., 1941, or any book on theory of numbers.

Write as the product of primes:

3. 10.
7. 72.
11. 648.
15. 98.
4. 8.
8. 81.
12. 90.
16. 490.
5. 40.
9. 162.
13. 270.
17. 2450.
6. 45.
10. 324.
14. 810.
18. 17,150.

11.4 Highest Common Factor

11.4.1 DEFINITION: The *highest common factor* of two (or more) numbers is the largest number which is a factor of each of the numbers. We abbreviate highest common factor HCF.

We already have had examples of the HCF of two numbers. We saw that 12 is the HCF of 24 and 60.

If the numbers are small, it is generally not difficult to pick out their HCF. But, if there is any difficulty, we can pick it out systematically by writing the numbers in terms of their prime factors. We then take the product of all different prime factors common to the given numbers, each taken the *smallest* number of times that it occurs in any of the numbers.

EXAMPLES 11.4.2

1. The HCF of 15 and 10 is 5, since $15 = 3 \cdot 5$ and $10 = 2 \cdot 5$, and the only common prime factor is 5.

2. The HCF of 24 and 60 is 12, since $24 = 2^3 \cdot 3$ and $60 = 2^2 \cdot 3 \cdot 5$. The common prime factors are 2^2 and 3 and $2^2 \cdot 3 = 12$. (2^2 means 2 is a prime factor twice.)

3. The HCF of 8 and 48 is 8, since $8 = 2^3$ and $48 = 2^4 \cdot 3$.

4. The HCF of 162 and 72 is $2 \cdot 3^2 = 18$, since $162 = 2 \cdot 3^4$ and $72 = 2^3 \cdot 3^2$.

5. If two numbers have no prime factors in common, the HCF is said to be 1. Thus the HCF of $40 = 2^3 \cdot 5$ and $63 = 3^2 \cdot 7$ is 1. Two such numbers are said to be *relatively prime.*

6. The HCF of 20, 150, and 120 is 10, since $20 = 2^2 \cdot 5$, $150 = 2 \cdot 3 \cdot 5^2$, $120 = 2^3 \cdot 3 \cdot 5$, and the least number of times 2 occurs as a factor is once (of 150) and the least number of times 5 occurs is once (of 20 and 120) and 3 does not occur at all as a factor of 20.

EXERCISE 11.4.3

Find the HCF of:

1. 45 and 75.
2. 63 and 72.
3. 90 and 100.
4. 90 and 300.
5. 98 and 42.
6. 210 and 88.
7. 630 and 420.
8. 42 and 210.
9. 45, 75, and 25.
10. 30, 140, 1100.
11. 2205, 2450, 1715.
12. 25725, 5775, 3675.
13. x^2y and xy^2.
14. $6ab^3$ and $9a^2b^2$.
15. $33p^2q^2r^2$ and $121pq^2r^3$.
16. $24x^2y$, $18x^3y^2$, $54x^4$.

Reduce to lowest terms:

17. $\frac{45}{75}$.
18. $\frac{98}{42}$.
19. $\frac{630}{420}$.
20. $\frac{153}{189}$.
21. $\dfrac{3+12}{4+21}$.
22. $\dfrac{2(3+5)+6}{3(17-7)+15}$.
23. $\dfrac{2^3\cdot5^2+2\cdot5\cdot7}{2^4\cdot5^3+2^2\cdot5\cdot11}$.
24. $\dfrac{3^2+2\cdot3^3+5\cdot3\cdot7}{3\cdot7+3^3\cdot2^2+2\cdot3^2}$.
25. $\dfrac{2^2+3^2+5^2}{3^2+5^2+7^2}$.
26. $\dfrac{2^2\cdot5^2\cdot7+3\cdot7+2^3\cdot5+3^2}{2^2[3(43-36)]}$.
27. $\dfrac{x^3}{x^5}$.
28. $\dfrac{6x^2y}{9xy^2}$.
29. $\dfrac{xy+x^2}{x+x^2}$.
30. $\dfrac{18x^2+24xy}{30x^3+42x^2y}$.
31. $\dfrac{(2x+2y)^2}{4(x-y)(x+y)}$.
32. $\dfrac{30x^2-30y^2}{25x^2+50xy+25y^2}$.

11.5 Least Common Multiple

Another number besides the HCF which depends upon two (or more) other numbers is the least common multiple. (See Example 7 of Examples 10.1.4.) This number also is useful in working with fractions.

11.5.1 **Definition:** The *least common multiple* (LCM) of two (or more) numbers is the smallest number which is a multiple of each of the numbers (or the smallest number of which each of the numbers is a factor or divisor).

Recall that the integer a is a multiple of the integer b if there exists an integer x such that $a = bx$ and that b is then a factor or divisor of a.

The LCM may be found by writing each number in terms of its prime factors and by taking the product of the different prime factors, each taken the *greatest* number of times that it occurs in any of the given numbers.

Examples 11.5.2

1. The LCM of 10 and 15 is 30, since $10 = 2 \cdot 5$ and $15 = 3 \cdot 5$. $\text{LCM} = 2 \cdot 5 \cdot 3 = 30$.
2. The LCM of 12 and 30 is 60, since $12 = 2^2 \cdot 3$ and $30 = 2 \cdot 3 \cdot 5$. $\text{LCM} = 2^2 \cdot 3 \cdot 5 = 60$.
3. The LCM of 90 and 12 is 180, since $90 = 2 \cdot 3^2 \cdot 5$ and $12 = 2^2 \cdot 3$. $\text{LCM} = 2^2 \cdot 3^2 \cdot 5$.
4. The LCM of $600 = 2^3 \cdot 3 \cdot 5^2$, $180 = 2^2 \cdot 3^2 \cdot 4$, and $1400 = 2^3 \cdot 5^2 \cdot 7$ is $2^3 \cdot 3^2 \cdot 5^2 \cdot 7 = 12{,}600$.
5. The LCM of 10 and 21 is 210. 10 and 21 are relatively prime.

Exercise 11.5.3

Find the LCM:

1. 20 and 6.
2. 40 and 100.
3. 90 and 60.
4. 225 and 945.
5. 450 and 945.
6. 225 and 1890.
7. 1575 and 270.
8. 20, 6, and 15.
9. 20, 6, and 21.
10. 40, 100, and 70.
11. 450, 945, and 100.
12. 64, 96, and 144.
13. 288, 540, and 500.
14. 24, 36, 54, and 81.
15. x, y.
16. x^2, xy.
17. x^2y, xy.
18. x^2y, xy^2.
19. x^3y, xy^3.
20. x^3y^2, x^2y^3.
21. x^5y^2, x^2y^5.
22. x^5y^3, xy^2.

23. $20x^2, 6y^2$.
24. $40x^3y, 100x^4y^2$.
25. $90x^3y^2, 60xy^5$.
26. $40x^3y, 100x^4y^2, 90x^3y^2$.
27. $30xy^2, 140x^3, 1100y^3$.
28. $6x + 6y, 9(x + y)^2$.

29. $18x^2 + 24xy, 30x^3 + 42x^2y$.
30. $(2x + 2y)^2, 8(x + y)(x - y)$.

11.6 Least Common Denominator

We shall learn soon that in order to add fractions it is necessary to write them with a common denominator. For instance, to add $1/6$ and $5/8$ we must first express each fraction with the same denominator. We like to use as small numbers as possible. The *least* common denominator is the LCM of the denominators. For $1/6$ and $5/8$ the LCM of the denominators is 24 and we may write

$$\frac{1}{6} = \frac{1 \cdot 4}{6 \cdot 4} = \frac{4}{24} \quad \text{and} \quad \frac{5}{8} = \frac{5 \cdot 3}{8 \cdot 3} = \frac{15}{24}.$$

The LCM of the denominators is called the least common denominator, LCD. Sometimes the fractions are not in lowest terms; they should first be written in lowest terms.

Exercise 11.6.1

Write with the LCD:

1. $\frac{1}{20}, \frac{5}{6}$.
2. $\frac{17}{90}, \frac{19}{60}$.
3. $\frac{2}{225}, \frac{8}{945}$.
4. $\frac{3}{20}, \frac{11}{6}$.
5. $\frac{7}{20}, \frac{5}{6}, \frac{31}{15}$.
6. $\frac{25}{16}, \frac{18}{32}$.
7. $\frac{21}{36}, \frac{25}{24}$.
8. $\frac{30}{15}, \frac{5}{8}$.
9. $\frac{27}{18}, \frac{15}{10}, \frac{17}{34}$.
10. $\frac{5}{24}, \frac{1}{36}, \frac{7}{54}, \frac{36}{81}$.
11. $\frac{1}{x}, \frac{1}{y}$.
12. $\frac{3}{x^2}, \frac{2}{xy}$.
13. $\frac{y}{x}, \frac{x}{y}$.
14. $\frac{2y}{3x}, \frac{3y}{2x}$.
15. $\frac{2y}{3x}, \frac{3x}{2y}$.
16. $\frac{7}{x^2}, \frac{5}{xy}$.
17. $\frac{2y}{x^2}, \frac{7y}{3x^2}$.
18. $\frac{3}{40x^2y}, \frac{15}{100xy^2}$.
19. $\frac{7}{40x^2y}, \frac{27}{300x^4y^2}, \frac{31}{90x^3y^2}$.
20. $\frac{90x^3}{2700x^4y^2}, \frac{63y}{140x^3y}, \frac{2x}{2200y^3}$.

21. $\dfrac{5}{6x + 6y}, \dfrac{7}{9(x + y)^2}.$ **22.** $\dfrac{x}{x + y}, \dfrac{y}{x - y}.$

23. $\dfrac{3x^2}{3x^2 + 4xy}, \dfrac{2y^2}{3xy + 4y^2}.$

24. $\dfrac{x - y}{(2x + 2y)^2}, \dfrac{x + y}{8(x + y)(x - y)}.$

11.7 Multiplication of Fractions

Fractions were introduced to us through the idea of dividing things into equal parts: $\frac{1}{3}$ means one of three equal parts, and in general a/b means a of b equal parts.

Suppose that we consider a square, 1 unit on a side. By dividing one side into $\frac{1}{5}$'s and the other into $\frac{1}{3}$'s we divide the square into 15 equal rectangles as shown on our figure. The area of each of these rectangles is $\frac{1}{15}$; that is, each is one of 15 equal parts into which the square has been divided. But the area of a rectangle equals its length times its width, so that for each rectangle we want $\frac{1}{3} \cdot \frac{1}{5} = \frac{1}{15}$. If we consider the shaded area, consisting of 6 rectangles, its area is $\frac{6}{15}$, 6 of 15 equal parts. But the length of this shaded area is $\frac{2}{3}$ (2 of 3 equal parts) and its width is $\frac{3}{5}$ (3 of 5 equal parts). We then want $\frac{2}{3} \cdot \frac{3}{5} = \frac{6}{15}$.

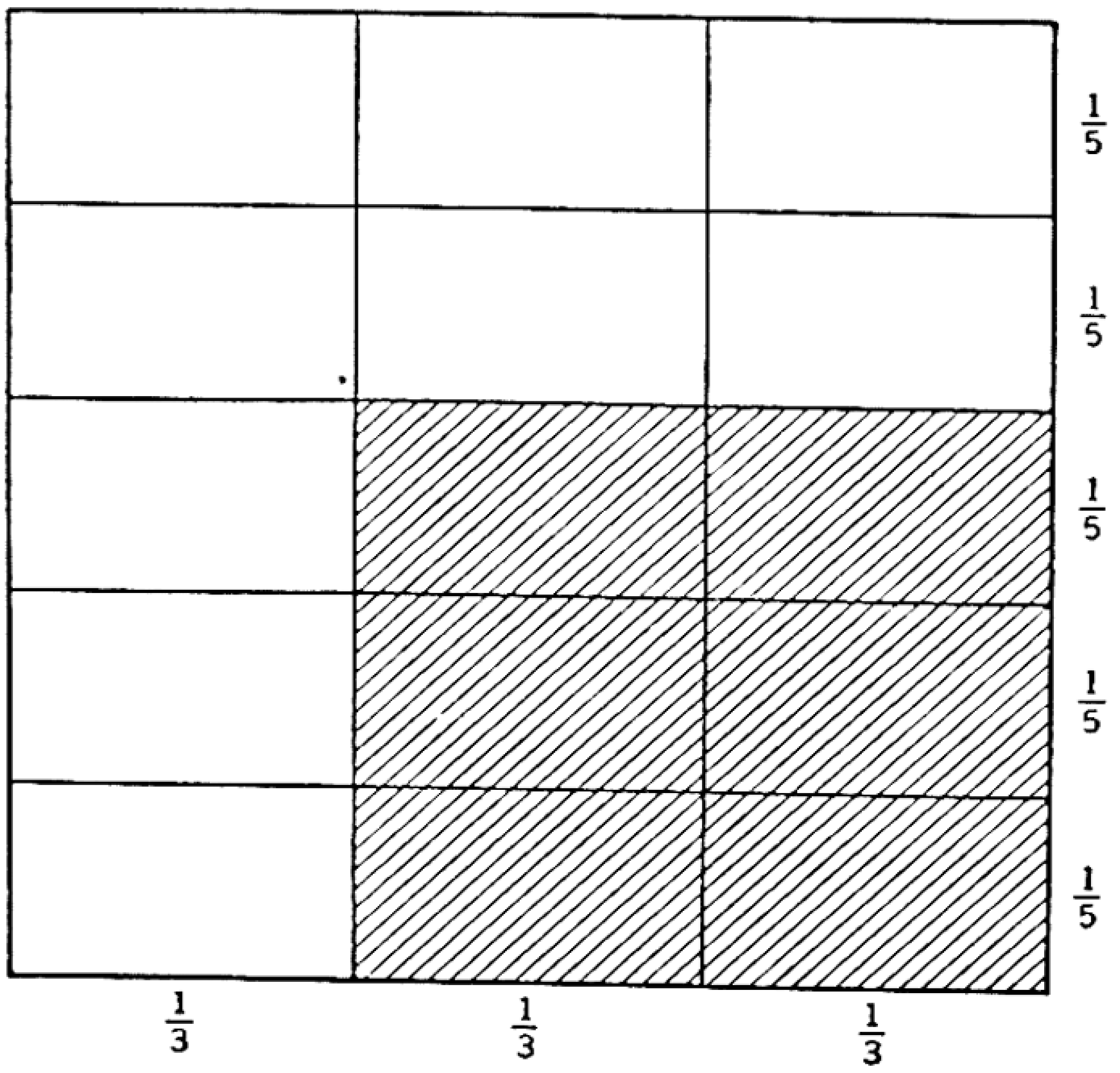


FIG. 11.7.1.

This suggests a definition for the product of two fractions.

11.7.2 DEFINITION: If a/b and c/d are fractions then $\frac{a}{b}\cdot\frac{c}{d} = \frac{ac}{bd}$.

In words: the product of two fractions is a fraction whose numerator is the product of the numerators of the given fractions and whose denominator is the product of the denominators of the given fractions.

EXAMPLES 11.7.3

1. $\frac{1}{2}\cdot\frac{1}{3} = \frac{1\cdot 1}{2\cdot 3} = \frac{1}{6}$.
2. $\frac{5}{2}\cdot\frac{3}{7} = \frac{15}{14}$.
3. $\frac{3}{5}\cdot\frac{2}{1} = \frac{6}{5}$.
4. $\frac{2}{7}\cdot\frac{9}{2} = \frac{18}{14} = \frac{9\cdot 2}{7\cdot 2} = \frac{9}{7}$.
5. $\frac{15}{7}\cdot\frac{14}{25} = \frac{3\cdot 5\cdot 2\cdot 7}{7\cdot 5^2} = \frac{(3\cdot 2)(5\cdot 7)}{5\cdot(5\cdot 7)} = \frac{6}{5}$.
6. $\frac{18}{10}\cdot\frac{35}{54} = \frac{2\cdot 3^2}{2\cdot 5}\cdot\frac{5\cdot 7}{2\cdot 3^2} = \frac{7\cdot(2\cdot 3^2\cdot 5)}{6\cdot(2\cdot 3^2\cdot 5)} = \frac{7}{6}$.
7. $(\frac{2}{3})^2 = \frac{2}{3}\cdot\frac{2}{3} = \frac{4}{9}$.
8. $(\frac{5}{2})^2 = \frac{25}{4}$.
9. $\frac{x}{y}\cdot\frac{u}{v} = \frac{xu}{yv}$.
10. $\frac{3x}{2y}\cdot\frac{x^2}{3y^3} = \frac{3x^3}{6y^4} = \frac{x^3}{2y^4}$.

EXERCISE 11.7.4

Multiply (reduce your answer to lowest terms):

1. $\frac{2}{3}\cdot\frac{4}{5}$. (Check this with the figure.)
2. $\frac{1}{3}\cdot\frac{3}{5}$.
3. $\frac{1}{5}\cdot\frac{2}{3}$.
4. $\frac{3}{7}\cdot\frac{3}{5}$.
5. $\frac{3}{7}\cdot\frac{5}{3}$.
6. $\frac{6}{11}\cdot\frac{4}{15}$.
7. $\frac{9}{8}\cdot\frac{20}{21}$.
8. $\frac{45}{98}\cdot\frac{42}{75}$.
9. $\frac{7}{1}\cdot\frac{1}{14}$.
10. $(\frac{2}{3}\cdot\frac{4}{5})\cdot\frac{25}{16}$.
11. $\frac{2}{3}\cdot(\frac{4}{5}\cdot\frac{25}{16})$.
12. $\frac{2}{3}\cdot\frac{4}{5}\cdot\frac{25}{16}$.
13. $\frac{15}{8}\cdot\frac{12}{35}\cdot\frac{14}{9}$.
14. $\frac{18}{35}\cdot\frac{125}{54}\cdot\frac{21}{55}\cdot\frac{56}{15}$.
15. $(\frac{7}{4})^2$.
16. $\frac{2}{3}\cdot(\frac{3}{4})^2$.
17. $(\frac{2}{3})^2\cdot\frac{3}{4}$.
18. $(\frac{2}{3})^2\cdot(\frac{3}{4})^2$.
19. $(\frac{2}{3}\cdot\frac{3}{4})^2$.
20. $(\frac{3}{5})^2\cdot(\frac{1}{2})^2$.
21. $(\frac{3}{5}\cdot\frac{1}{2})^2$.
22. $\frac{x}{y}\cdot\frac{x^2}{1}$.

23. $\left(\frac{x}{y}\right)^2$. 24. $\left(\frac{x}{y}\right)^2 \cdot \frac{x}{y}$. 25. $\left(\frac{x}{y}\right)^3$.

26. $\left(\frac{2x}{y}\right)^2$. 27. $\left(\frac{x^2}{y}\right)^2$. 28. $\left(\frac{x^2}{y^2}\right)^2$.

29. $\left(\frac{x^2}{y^3}\right)^3$. 30. $\left(\frac{x^2}{y^3}\right)^2 \cdot \left(\frac{x^3}{y^2}\right)^2$. 31. $\left(\frac{x^2}{y^3} \cdot \frac{x^3}{y^2}\right)^2$.

32. $\frac{3x}{2y} \cdot \frac{8xy}{9}$. 33. $\frac{18ab}{25cd} \cdot \frac{10ac}{27bd}$. 34. $\frac{100ab}{63c^2} \cdot \frac{7ac}{16b}$.

35. $\frac{36a^2bx}{81by^3z} \cdot \frac{27ay^3z^2}{12a^3xz}$. 36. $\frac{t+1}{1} \cdot \frac{1}{t^2+t}$.

37. $\frac{4xy}{x+3} \cdot \frac{3x^2+9x}{16y^2}$. 38. $\frac{14x^2-7x}{12x^3+24x^2} \cdot \frac{x^2+2x}{2x-1}$.

39. $\frac{3ax^2-9ax}{10x^2+5x} \cdot \frac{2x^3+x^2}{a^2x-3a^2}$. 40. $\frac{x^2y^3-xy^4}{x^4y^3+x^3y^4} \cdot \frac{x^3y^3+x^2y^4}{x^3y-x^2y^2}$.

We may note at this point that *the system of fractions is closed under the operation of multiplication.* For if $\frac{a}{b}$ and $\frac{c}{d}$ are two fractions their product is $\frac{ac}{bd}$ and ac and bd are natural numbers (since the system of natural numbers is closed under multiplication); $\frac{ac}{bd}$ is therefore a fraction.

11.8 Fractions and Division

We have pointed out that fractions were introduced to us through the idea of dividing things into equal parts. We now identify the fraction $\frac{a}{b}$ with the quotient $a \div b$. We *define*

11.8.1 $$\frac{a}{b} = a \div b.$$

This gives a meaning to $a \div b$ where a and b are any natural numbers. Previously $a \div b$ had a meaning only if a was a multiple of b or b was a factor or divisor of a.

EXAMPLES 11.8.2

1. $\frac{1}{3} = 1 \div 3$.
2. $\frac{2}{3} = 2 \div 3$.
3. $\frac{5}{2} = 5 \div 2$.
4. We have previously stated that $\frac{5}{0}$ is not a fraction. We now see another reason for this. If $\frac{5}{0}$ were a fraction we should want to identify it with $5 \div 0$. But we may never divide by zero. $a/0$ is not a fraction for any value of a.
5. $\frac{6}{3} = 6 \div 3$, but $6 \div 3 = 2$ and $\frac{6}{3} = \frac{2}{1}$. We should therefore identify the fraction $\frac{2}{1}$ with the natural number 2.
6. $\frac{15}{3} = 15 \div 3$ and therefore $\frac{5}{1} = 5$.
7. $\frac{1}{1} = 1 \div 1 = 1$.
8. $a = a/1$ if a is a natural number. Thus a natural number a may always be considered "in fractional form" as $a/1$.

This idea of identifying a/b with $a \div b$ is also suggested by our figures 11.2.1 and 11.2.2. We had found it convenient to label with natural numbers points along a line placed at equal distances from a beginning point. The beginning point was labeled 0. (This "beginning point" is frequently called the *origin*.)

It seems sensible to label the "fracture points" with fractions. Of course, we should get into trouble if we actually tried to put a label on every fracture point. For instance, $49/100$ is very close to $1/2$; $499/1000$ is still closer to $1/2$. There is no end to the fractions we can place between $49/100$ and $1/2$, and if we tried to label all the points corresponding to these fractions we should have a mess. (Can you find some fractions closer to $1/2$ than $499/1000$?)[19]

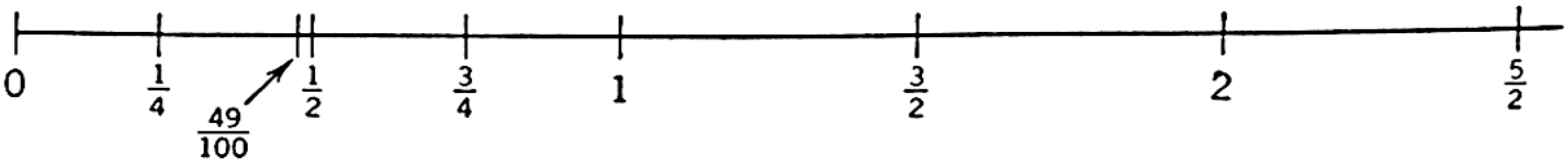


FIG. 11.8.3.

11.8.4 DEFINITION: Two numbers are said to be *reciprocals* of each other if their product is 1.

EXAMPLES 11.8.5

1. Since $\frac{2}{3}\cdot\frac{3}{2} = \frac{6}{6} = \frac{1}{1} = 1$, $\frac{2}{3}$ and $\frac{3}{2}$ are reciprocals of each other; $\frac{3}{2}$ is the reciprocal of $\frac{2}{3}$, and $\frac{2}{3}$ is the reciprocal of $\frac{3}{2}$.

2. $\frac{17}{8}$ and $\frac{8}{17}$ are reciprocals.

3. a/b and b/a are reciprocals; that is, the reciprocal of a given fraction is obtained by interchanging the numerator and denominator.

4. 2 and $\frac{1}{2}$ are reciprocals since $2\cdot\frac{1}{2} = \frac{2}{1}\cdot\frac{1}{2} = 1$.

5. If a is a natural number, its reciprocal is $1/a$.

6. 0 has no reciprocal.

11.9 Division of Fractions

11.9.1 DEFINITION: If a/b and c/d are fractions, then the quotient of a/b and c/d, in that order, means a fraction x/y such that $\frac{x}{y}\cdot\frac{c}{d} = \frac{a}{b}\cdot$ The quotient is written $\frac{a}{b} \div \frac{c}{d}$, so that $\frac{x}{y} = \frac{a}{b} \div \frac{c}{d}$; $\frac{a}{b}$ is called the dividend, and $\frac{c}{d}$ is called the divisor.

EXAMPLES 11.9.2

1. $\frac{2}{3} \div \frac{5}{7}$ means a fraction $\frac{x}{y}$ such that $\frac{x}{y}\cdot\frac{5}{7} = \frac{2}{3}\cdot$ In this case $\frac{x}{y} = \frac{2}{3}\cdot\frac{7}{5} = \frac{2\cdot 7}{3\cdot 5} = \frac{14}{15}$ since $\frac{14}{15}\cdot\frac{5}{7} = \frac{2\cdot 7\cdot 5}{3\cdot 7\cdot 5} = \frac{2}{3}\cdot$ Thus $\frac{2}{3} \div \frac{5}{7} = \frac{14}{15}$.

2. $\frac{4}{5} \div \frac{2}{7}$ means a fraction $\frac{x}{y}$ such that $\frac{x}{y}\cdot\frac{2}{7} = \frac{4}{5}\cdot$ In this case $\frac{x}{y} = \frac{4\cdot 7}{5\cdot 2} = \frac{28}{10} = \frac{14}{5}$ since $\frac{14}{5}\cdot\frac{2}{7} = \frac{2\cdot 2\cdot 7}{5\cdot 7} = \frac{4}{5}\cdot$ Thus $\frac{4}{5} \div \frac{2}{7} = \frac{14}{5}$.

3. $\frac{4}{9} \div \frac{10}{21} = \frac{x}{y}$ such that $\frac{x}{y} \cdot \frac{10}{21} = \frac{4}{9}$. In this case

$\frac{x}{y} = \frac{84}{90} = \frac{2^2 \cdot 3 \cdot 7}{3^2 \cdot 2 \cdot 5} = \frac{2 \cdot 7 \cdot (2 \cdot 3)}{3 \cdot 5 \cdot (2 \cdot 3)} = \frac{14}{15}$ since $\frac{14}{15} \cdot \frac{10}{21} = \frac{2^2 \cdot 5 \cdot 7}{3^2 \cdot 5 \cdot 7} = \frac{4}{9}$.

4. $\frac{5}{32} \div \frac{15}{16} = \frac{5 \cdot 16}{32 \cdot 15} = \frac{5 \cdot 2^4}{3 \cdot 5 \cdot 2^5} = \frac{1}{3 \cdot 2} = \frac{1}{6}$ since $\frac{15}{16} \cdot \frac{1}{6} = \frac{5}{32}$.

5. $\frac{14}{25} \div \frac{2}{15} = \frac{21}{5}$. Show that this is correct.

Study of these examples indicates that there is a simple rule for finding the quotient of two fractions:

11.9.3 *The quotient of two fractions is equal to the product of the dividend and the reciprocal of the divisor.*

In symbols,

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}.$$

For $\frac{a}{b} \div \frac{c}{d}$ means a fraction $\frac{x}{y}$ such that $\frac{x}{y} \cdot \frac{c}{d} = \frac{a}{b}$. But, if $\frac{x}{y} = \frac{ad}{bc}$, then

$$\frac{x}{y} \cdot \frac{c}{d} = \frac{ad}{bc} \cdot \frac{c}{d} = \frac{a(cd)}{b(cd)} = \frac{a}{b}.$$

We have previously shown that the product of two fractions is a fraction. We see that *the system of fractions is closed under the operation of division* since we obtain the quotient by multiplying the dividend by the reciprocal of the divisor, in other words, by multiplication of two fractions.

EXERCISE 11.9.4

1. What is meant by the last italicized statement above?

2. If p is a natural number and $\frac{r}{s}$ a fraction, how would you find $p \div \frac{r}{s}$?

3. How would you find $\frac{r}{s} \div p$?

Perform the indicated operations (reduce your answers to lowest terms):

4. $\frac{2}{3} \div \frac{5}{4}$.
5. $\frac{1}{3} \div \frac{5}{3}$.
6. $\frac{1}{5} \div \frac{3}{2}$.
7. $\frac{3}{7} \div \frac{5}{3}$.
8. $\frac{3}{7} \div \frac{3}{5}$.
9. $\frac{6}{11} \div \frac{15}{4}$.
10. $\frac{9}{8} \div \frac{21}{20}$.
11. $\frac{45}{98} \div \frac{75}{42}$.
12. $3 \div \frac{3}{5}$.
13. $3 \div \frac{5}{3}$.
14. $\frac{3}{5} \div 3$.
15. $\frac{5}{3} \div 3$.

16. $\frac{2}{3} \cdot (\frac{5}{8} \div \frac{10}{9})$.
17. $(\frac{2}{3} \cdot \frac{5}{8}) \div \frac{10}{9}$.
18. $(\frac{2}{3} \div \frac{5}{8}) \cdot \frac{10}{9}$.
19. $\frac{2}{3} \div (\frac{5}{8} \cdot \frac{10}{9})$.
20. $(\frac{3}{4} \div \frac{5}{8}) \cdot (\frac{10}{9} \div \frac{2}{3})$.
21. $\frac{3}{4} \div [\frac{5}{8} \cdot (\frac{10}{9} \div \frac{2}{3})]$.
22. $[(\frac{3}{4} \div \frac{5}{8}) \cdot \frac{10}{9}] \div \frac{2}{3}$.
23. $(\frac{1}{2} \div \frac{1}{3}) \div \frac{1}{4}$.
24. $\frac{1}{2} \div (\frac{1}{3} \div \frac{1}{4})$.
25. $\frac{36}{25} \div \frac{54}{15}$.
26. $\frac{60}{847} \div \frac{72}{539}$.

Perform the indicated operations. Find the value of the answer for the given values of the letters. Check your result by substituting the values of the letters in the original expression and combining the resulting numbers.

27. $\frac{x}{y} \div \frac{y}{x}$; $x = 2$, $y = 3$.
28. $\frac{a}{b} \div \frac{a}{b}$; $a = 2$, $b = 3$.
29. $\frac{x^2}{y} \div \frac{y^2}{x}$; $x = 5$, $y = 3$.
30. $\frac{x^2}{y} \div \frac{x^2}{y^2}$; $x = 5$, $y = 3$.
31. $\frac{x^2}{y^2} \div xy$; $x = 3$, $y = 5$.
32. $\left(\frac{2a}{b^2} \cdot \frac{c}{d^2}\right) \div \frac{6c^2}{bd^2}$; $a = b = 2$, $c = d = 3$.
33. $\frac{2a}{b^2} \cdot \left(\frac{c}{d^2} \div \frac{6c^2}{bd^2}\right)$; $a = b = 2$, $c = d = 3$.
34. $\left(\frac{2a}{b^2} \div \frac{c}{d^2}\right) \cdot \frac{6c^2}{bd^2}$; $a = b = 2$, $c = d = 3$.
35. $\frac{2a}{b^2} \div \left(\frac{c}{d^2} \cdot \frac{6c^2}{bd^2}\right)$; $a = b = 2$, $c = d = 3$.
36. $\frac{57abx^3}{87cdy} \div \frac{76a^2bx^2}{58cdy^2}$; $a = b = c = d = 3$, $x = y = 5$.
37. $\frac{86a^2b^3c}{17m^2n^2p} \div \frac{129abc^2}{34mn^2p}$; $a = b = c = 3$, $m = n = p = 4$.

38. $\dfrac{ax - a}{bx - 3b} \div \dfrac{cx - c}{dx - 3d}$; $a = 1, b = 2, c = 3, d = 4, x = 5$.

39. $\dfrac{7x + 7}{9x - 9} \div \dfrac{14x + 14}{3x - 3}$; $x = 2$.

40. $\dfrac{35a - 25}{12a + 44} \div \dfrac{28a - 20}{36a + 132}$; $a = 2$.

The set of integers is not closed under the operation of division, but the set of fractions is closed under this operation. There is no integer x such that $5x = 3$, and therefore $3 \div 5$ is not an integer. But there is a fraction x such that $5x = 3$. For, if $x = 3/5$, $5 \cdot \dfrac{3}{5} = \dfrac{5}{1} \cdot \dfrac{3}{5} = \dfrac{3 \cdot 5}{1 \cdot 5} = \dfrac{3}{1} = 3$. In general, if we have an equation $ax = b$, then, by definition of division, $x = b \div a = b/a$. Of course, if b is a multiple of a, then $\dfrac{b}{a}$ is an integer.*

EXERCISE 11.9.4 (continued)

Solve the following equations:

41. $5x = 3$.
42. $9x = 12$.
43. $3x = 15$.
44. $15x = 3$.
45. $3x + 8 = 23$.
46. $3x + 4 = 23$.
47. $2t + 1 = 2$.
48. $5x + 13 = 33$.
49. $5s + 2 = 33$.
50. $5(m + 3) = 35$.
51. $5(m + 3) = 32$.
52. $6(t + 2) = 26$.
53. $6(t + 2) = 24$.
54. $3y + 2(y + 1) = 27$.
55. $3y + 2(y + 1) = 61$.
56. $5x - 15 = 0$.
57. $5x - 19 = 0$.
58. $14a - 21 = 0$.
59. $14a - 24 = 0$.
60. $121p - 132 = 0$.
61. $121p - 95 = 0$.
62. $(3x - 15)(5x - 15) = 0$.
63. $(3x - 7)(5x - 9) = 0$.
64. $(2q - 18)(5q - 7) = 0$.
65. $(2q - 11)(5q - 25) = 0$.
66. $(14s - 25)(14s - 31) = 0$.
67. $(14s - 21)(14s - 24) = 0$.
68. $(2x - 1)(3x - 1)(4x - 1) = 0$.
69. $(5x - 3)(7x - 35)(12x - 42) = 0$.

---

* We might also say that we solve this equation by dividing the expressions on each side of the = sign by a. For then we have $\dfrac{ax}{a} = \dfrac{b}{a}$ or $x = \dfrac{b}{a}$.

70. $4x - 2 = 6.$
71. $4x - 2 = 11.$
72. $18x - 7 = 51.$
73. $n + (n + 1) + (n + 2) = 18.$
74. $n + (n + 1) + (n + 2) = 20.$

For what values of x are the following meaningless?

75. $\frac{5}{x - 3}.$
76. $\frac{-3}{x + 4}.$
77. $\frac{2}{5x - 3}.$
78. $\frac{x}{(3x - 15)(5x - 15)}.$
79. $\frac{x - 2}{x - 2}.$
80. $\frac{0}{x - 7}.$
81. $\frac{x + 1}{x - 1}.$
82. $\frac{x + 1}{x^2}.$
83. $\frac{x - 1}{x^2 + 1}.$
84. $\frac{x - 1}{x^2 - 1}.$
85. $\frac{3}{9 - x^2}.$
86. $\frac{3}{(9 - x)^2}.$
87. $\frac{9 - x^2}{(9 - x)^2}.$
88. $\frac{x - 4}{(x - 1)(x - 2)(x - 3)}.$
89. $\frac{(x - 1)(x - 2)(x - 3)}{x - 4}.$
90. $\frac{x^2 + x + 1}{(2x - 1)(3x + 2)(4x - 3)(5x + 4)}.$

11.10 Problems and Their Solutions

We have had several sets of "word problems." You may not have considered the matter, but in each such problem that we have had thus far the answer obtained made sense for that problem. For instance, in Problem 76 of Exercise 10.2.2 we were asked, "What three consecutive natural numbers have a sum of 18?" If you set up the equation correctly, you obtained $n + (n + 1) + (n + 2) = 18$. When we solve this equation we obtain $n = 5$ so that the three wanted numbers are 5, 6, and 7. Now, if the problem had read, What three consecutive natural numbers have a sum of 20?, most of us would have set to work and obtained the equation $n + (n + 1) + (n + 2) = 20$. On the face of it, these problems appear quite similar. But when we solve this last equation we obtain $n = 1\frac{7}{3}$, which is not a natural number at all! The answer to our second question, What three consecutive natural numbers have a sum of 20? is, *there are no such numbers.* When a problem is proposed we may not know beforehand whether there is a solution to it. (This statement applies not only to mathematics but to other fields.) But mathematics often gives us a procedure which simplifies the attack upon a problem *whether or not the problem has a solution.*

If we consider the more general question, What three consecutive natural numbers have a sum of s, where s is a natural number?, a little thought will reveal that there are three such natural numbers if and only if s is greater than 3 and divisible by 3. We obtain an answer to our question if s is 6, 9, 12, 15, $\cdots$ but not otherwise. In a certain sense there are twice as many values of s for which we cannot find three such numbers as there are values for which we can find them.

For the example we have considered there is a solution of the *equation* $n + (n + 1) + (n + 3) = 20$. We can find a number of our system (the fractions) which makes the left side equal the right side. Later we shall have examples for which there are not even solutions to the equations. We shall discover that we cannot write down just any equation and expect in all cases to find a number, any number, which makes the left side of the equation equal the right side.

EXERCISE 11.10.1

Set up an equation for each problem and solve the equation. State whether the solution to the equation makes sense for the problem.

1. If twice a certain natural number is increased by 1 we obtain 43.
2. If twice a certain natural number is increased by 1 we obtain 38.
3. If twice a certain number is increased by 1 we obtain 38.
4. The sum of a certain natural number and three times that number is 56.
5. The sum of a certain natural number and three times that number is 42.

6. A piece of cord 42 feet long is cut so that one piece is three times as long as the other.

7. The sum of a certain natural number and twice that number is 42.

8. The sum of a certain natural number and twice that number is 32.

9. A piece of cord 32 feet long is cut so that one piece is twice as long as the other.

10. The sum of a certain number and twice that number is 32.

11. We have three natural numbers of which the first is twice the second and the third is four more than the first. Their sum is 59.

12. A cord 59 inches long is cut into three pieces of which the first is twice as long as the second and the third is 4 inches longer than the first.

13. A cord 41 inches long is cut into three pieces of which the first is twice as long as the second and the third is 4 inches longer than the first.

14. We have three natural numbers of which the first is twice the second and the third is four more than the first. Their sum is 41.

15. We have three numbers of which the first is twice the second and the third is four more than the first. Their sum is 41.

16. Dick has twice as many marbles as Tom, and Sam has four more than Dick. Together they have 41 marbles.

17. One angle of a triangle is 30°, and the second is twice the third. What are the angles?

18. One angle of a triangle is 35°, and the second is twice the third. What are the angles?

19. The length of a rectangle is twice its width. What are its dimensions if its perimeter is 66 feet?

20. The length of a rectangle is twice its width. What are its dimensions if its perimeter is 70 feet?

21. The number of square feet in the area of a rectangle is the same as the number of feet in its perimeter. What is its length if its width is 5 feet?

22. A, B, and C are angles of a triangle. Angle A is twice as large as angle B, and angle B is twice as large as angle C. What are the angles?

23. A, B, and C are angles of a triangle. Angle A is twice as large as angle B, and angle B is three times as large as angle C. What are the angles?

24. A, B, and C are angles of a triangle. Angle A is four times as large as angle B, and angle B is twice as large as angle C. What are the angles?

25. Roy, Albert, and Russell together have 180 marbles. Roy has four times as many as Albert, and Albert has twice as many as Russell. How many do each have?

12

Fractions; Rational Numbers

12.1 Addition of Fractions

We have interpreted the fraction a/b to mean a of b equal parts. If we have the fractions a/b and c/b and want to *add* them, we are asking what should be the sum of a of b equal parts and c of b equal parts. Certainly we should want this to be $(a + c)$ of b equal parts.

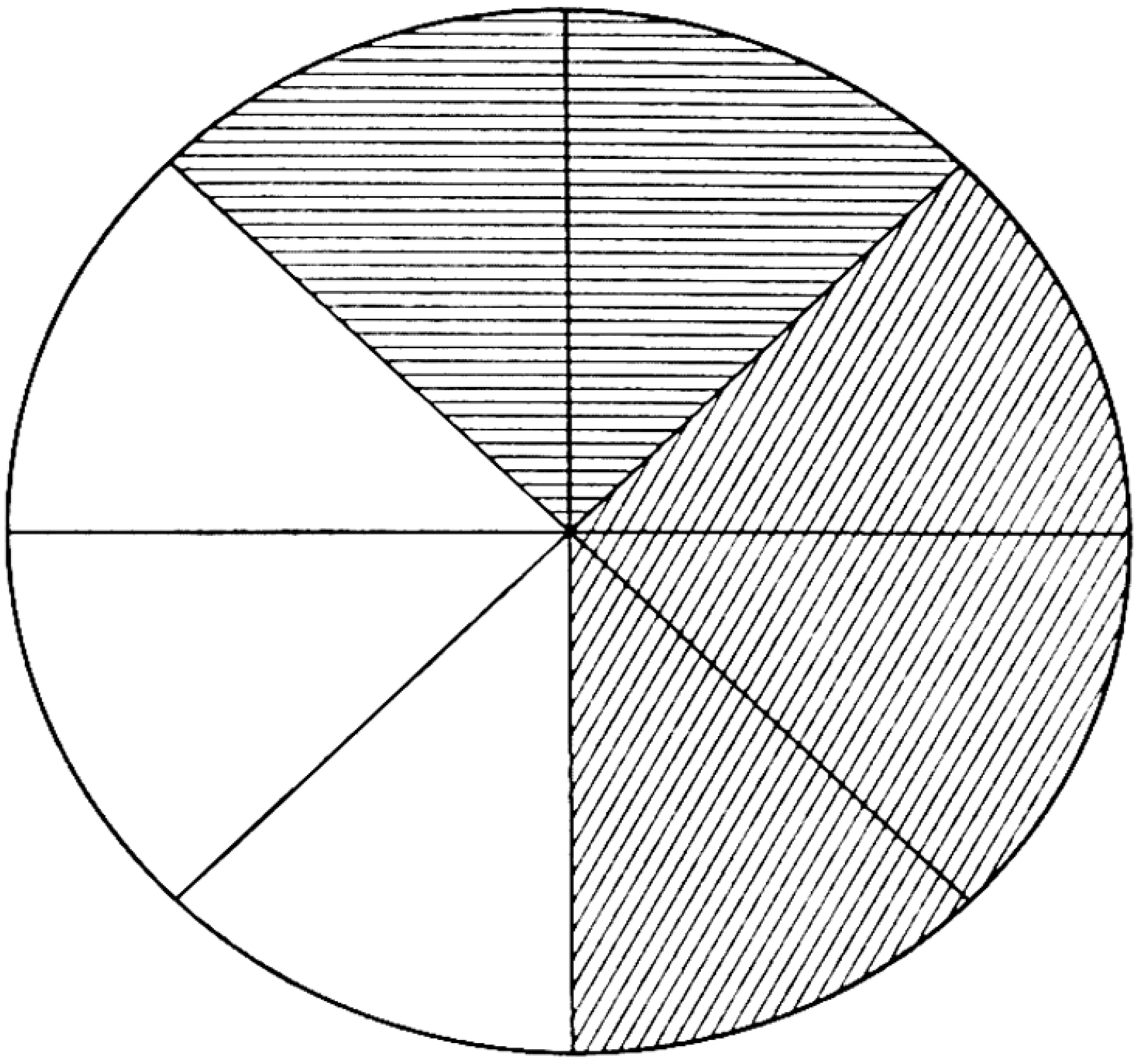


FIG. 12.1.1. $\frac{2}{8} + \frac{3}{8} = \frac{5}{8}$

12.1.2 DEFINITION: If $\frac{a}{b}$ and $\frac{c}{b}$ are fractions, then

$$\frac{a}{b} + \frac{c}{b} = \frac{a + c}{b}.$$

In words, *the sum of two fractions having the same denominator is a fraction whose denominator is this denominator and whose numerator is the sum of the given numerators.*

EXAMPLES 12.1.3

1. $\frac{2}{5} + \frac{4}{5} = \frac{6}{5}.$
2. $\frac{7}{2} + \frac{9}{2} = \frac{16}{2} = \frac{8}{1} = 8.$
3. $\dfrac{x}{z} + \dfrac{y}{z} = \dfrac{x + y}{z}.$
4. $\dfrac{a}{a + b} + \dfrac{b}{a + b} = \dfrac{a + b}{a + b} = \dfrac{1}{1} = 1.$

We have seen that we can always express two fractions which do not have a common denominator as two equal fractions having a common denominator. We had the example $1/6$ and $5/8$. Both of these can be expressed with denominator 24:

$$\frac{1}{6} = \frac{4}{24} \quad\text{and}\quad \frac{5}{8} = \frac{15}{24}.$$

We may then write:

$$\frac{1}{6} + \frac{5}{8} = \frac{4}{24} + \frac{15}{24} = \frac{19}{24}.$$

12.1.4 DEFINITION: The sum of two fractions with different denominators is the fraction obtained by adding two fractions, one equal to one of the given fractions and the other equal to the other given fraction, and having a common denominator.

In applying this definition it is usually best to use the LCD.

EXAMPLES 12.1.5

1. $\dfrac{2}{3} + \dfrac{3}{4} = \dfrac{2 \cdot 4}{3 \cdot 4} + \dfrac{3 \cdot 3}{4 \cdot 3} = \dfrac{8}{12} + \dfrac{9}{12} = \dfrac{17}{12}.$
2. $\frac{21}{36} + \frac{25}{24} = \frac{7}{12} + \frac{25}{24} = \frac{14}{24} + \frac{25}{24} = \frac{39}{24}.$
3. $\dfrac{1}{x} + \dfrac{1}{y} = \dfrac{y}{xy} + \dfrac{x}{xy} = \dfrac{y + x}{xy}.$
4. $\dfrac{7}{2x^2} + \dfrac{5}{3xy} = \dfrac{7 \cdot 3y}{6x^2y} + \dfrac{5 \cdot 2x}{6x^2y} = \dfrac{21y + 10x}{6x^2y}.$

We have previously noted that fractions are closed under the operations of multiplication and division. It is now apparent that *fractions are closed under the operation of addition.*

EXERCISE 12.1.6

Add (reduce answers to lowest terms):

1. $\frac{7}{16} + \frac{1}{16}; \frac{a}{b} + \frac{1}{b}.$
2. $\frac{19}{24} + \frac{23}{24}; \frac{x}{z} + \frac{y}{z}.$
3. $\frac{19}{3} + \frac{5}{3}; \frac{p}{q} + \frac{r}{q}.$
4. $\frac{3}{8} + \frac{7}{8} + \frac{15}{8}; \frac{a}{d} + \frac{b}{d} + \frac{c}{d}.$
5. $\frac{3}{7} + \frac{5}{7}; \frac{3}{x} + \frac{5}{x}.$
6. $\frac{2}{9} + \frac{5}{9}; \frac{x}{y^2} + \frac{z}{y^2}.$
7. $\frac{21}{25} + \frac{28}{25}; \frac{3a}{5b} + \frac{4a}{5b}.$
8. $\frac{8}{21} + \frac{20}{21}; \frac{4t}{21s} + \frac{10t}{21s}.$
9. $\frac{6}{5} + \frac{6}{5}; \frac{a}{b} + \frac{a}{b}.$
10. $\frac{9}{7} + \frac{12}{7}; \frac{3x}{x + 2y} + \frac{6y}{x + 2y}.$
11. $\frac{15p^2}{25p^3q + 15p^2q^2} + \frac{9pq}{25p^3q + 15p^2q^2}.$
12. $\frac{2}{3} + \frac{5}{2}; \frac{a}{b} + \frac{c}{a}.$
13. $\frac{1}{20} + \frac{4}{5}; \frac{1}{ax} + \frac{a}{x}.$ ✓
14. $\frac{3}{14} + \frac{7}{10}; \frac{a}{2x} + \frac{x}{2y}.$ ✓
15. $\frac{2}{3} + 2; \frac{a}{b} + a.$
16. $\frac{15}{8} + 4; \frac{ab}{2c} + c.$
17. $\frac{25}{16} + \frac{18}{32}; \frac{a^2}{b^4} + \frac{9b}{b^5}.$
18. $\frac{17}{14} + \frac{16}{28}; \frac{a}{7b} + \frac{b^4}{7b^2}.$

Add and reduce answers to lowest terms. Evaluate the answers for the given values of the letters; check by substituting the values of the letters in the original expressions and then performing the additions.

19. $\frac{2}{a} + \frac{a}{2}; a = 2.$
20. $\frac{x}{3} + \frac{x}{6}; x = 3.$

21. $\dfrac{a}{10} + \dfrac{b}{15}$; $a = 2, b = 5.$ 22. $\dfrac{1}{2x} + \dfrac{2}{3x}$; $x = 4.$

23. $\dfrac{a}{2x} + \dfrac{b}{3x}$; $a = 1, b = 2, x = 4.$

24. $\dfrac{2t}{5s} + \dfrac{5t}{2s}$; $t = 2, s = 5.$

25. $\dfrac{3x}{14y} + \dfrac{2x}{7y^2}$; $x = 7, y = 2.$

26. $\dfrac{2a}{3b} + \dfrac{3b}{2a}$; $a = 3, b = 2.$

27. $\dfrac{5p}{3x} + \dfrac{5q}{12y}$; $p = q = 1, x = y = 2.$

28. $\dfrac{3t + 1}{st} + \dfrac{3s + 1}{s^2}$; $s = t = 2.$

29. $\dfrac{x + 3}{x^2y} + \dfrac{x + 1}{xy^2}$; $x = 1, y = 3.$

30. $\dfrac{3}{40x^2y} + \dfrac{15}{100xy^2}$; $x = 3, y = 1.$

Perform the indicated operations (reduce answers to lowest terms):

31. $(4 + \frac{2}{3})(7 + \frac{1}{2}).$ 32. $(9 + \frac{3}{4})(2 + \frac{2}{3}).$

33. $(6 + \frac{2}{3}) \div (1 + \frac{5}{6}).$ 34. $(\frac{2}{9} + \frac{1}{3}) \div (\frac{3}{4} + \frac{1}{2}).$

35. $\left(\dfrac{a}{b} + b\right)\left(\dfrac{a}{b} + a\right).$ 36. $\left(c + \dfrac{d}{x}\right) \div \left(c + \dfrac{d}{x}\right).$

37. $\left(c + \dfrac{d}{x}\right)\left(c + \dfrac{d}{x}\right).$ 38. $\left(\dfrac{1}{x} + \dfrac{1}{y}\right)\left(\dfrac{y}{x} + \dfrac{x}{y}\right).$

39. $\left(\dfrac{x^2}{y^2} + \dfrac{x}{y}\right)\left(\dfrac{1}{x} + \dfrac{1}{x + y}\right).$

40. $\left(\dfrac{5a^2}{b^2} + \dfrac{2a}{b}\right)\left(\dfrac{1}{a} + \dfrac{b}{5a + 2b}\right).$

41. $\left(\dfrac{a}{b} + \dfrac{c}{d}\right) \div \left(\dfrac{a}{d} + \dfrac{bc}{d^2}\right).$

42. $\left(\frac{3t}{s}+\frac{5t}{s^2}\right) \div \left(\frac{3t}{s^2}+\frac{5t}{s^3}\right).$

43. $\left(y+3+\frac{10}{y}\right) \div \left(y+1+\frac{20}{y}\right).$

44. $\left(x+4+\frac{21}{x}\right) \div \left(x+8+\frac{7}{x}\right).$

45. $\dfrac{\frac{b+a}{ab}}{\frac{1}{a}}.$

46. $\dfrac{\frac{x^2+y^2}{xy}}{\frac{1}{x}}.$

47. $\dfrac{\frac{1}{a}+\frac{1}{b}}{\frac{1}{a}}.$

48. $\dfrac{\frac{x}{y}+\frac{y}{x}}{\frac{1}{x}}.$

49. $\dfrac{\frac{a}{b}+\frac{c}{d}}{\frac{a}{d}+\frac{bc}{d^2}}.$

For what values of x are each of the following meaningless?

50. $\frac{1}{x-1}+\frac{2}{x-2}.$

51. $\frac{x}{x+2}+\frac{5}{x+2}.$

52. $\frac{x+1}{x-1}+\frac{x-1}{x+1}.$

53. $\frac{1}{x}+\frac{2}{x+1}+\frac{3}{x+2}.$

54. $\frac{x^2+3}{(x+5)(x-2)}+\frac{x-1}{(x-2)(x-3)}.$

55. $\frac{x-2}{x-2}+\frac{7}{x+2}.$

56. $\frac{x-1}{(x+1)(x-1)}+\frac{x+1}{(x+1)(x+2)}.$

57. $\frac{5}{(x-3)^2}+\frac{4}{(x-3)(x-2)}.$

12.2 Complex Fractions

Problems 45 through 49 of Exercise 12.1.6 may seem strange to you. They are not fractions according to Definition 11.1.2. But we have identified the fraction a/b with the quotient $a \div b$, and it seems consistent now to identify a symbol as in Problem 45 with the quotient

$$\left(\frac{b+a}{ab}\right) \div \left(\frac{1}{a}\right).$$

Since fractions are closed under the operation of division we see that the symbol of Problem 45 then equals a fraction (if the letters represent natural numbers). It is called a complex fraction. A complex fraction is a fraction having one or more fractions in numerator or denominator or both.

It is usually inconvenient to work with a complex fraction, and we want to rewrite it as an equal *simple fraction*, a fraction having a natural number as numerator and a natural number as denominator. One way to accomplish this is to treat the complex fraction as a quotient.

EXAMPLES 12.2.1

1. $$\frac{\dfrac{x}{y}+\dfrac{y}{x}}{\dfrac{1}{x}} = \left(\frac{x}{y}+\frac{y}{x}\right) \div \frac{1}{x}$$

$$= \left(\frac{x^2+y^2}{xy}\right) \div \frac{1}{x}$$

$$= \left(\frac{x^2+y^2}{xy}\right) \cdot \frac{x}{1}$$

$$= \frac{x^2+y^2}{y}.$$

Another, frequently simpler, method is to make use of the principle that we obtain an equal fraction if we multiply both numerator and denominator of a fraction by the same number (not zero).

2. $$\frac{\dfrac{x^2+y^2}{xy}}{\dfrac{1}{x}} = \frac{\left(\dfrac{x^2+y^2}{xy}\right)\cdot xy}{\left(\dfrac{1}{x}\right)\cdot xy} = \frac{x^2+y^2}{y}.$$

As you probably expected, the fractions are to obey the associative, commutative, and distributive laws. The distributive law is of special importance in applying this second method of changing a complex fraction to an equal simple fraction.

3. $$\frac{\dfrac{x}{y} + \dfrac{y}{x}}{\dfrac{1}{x}} = \frac{\left(\dfrac{x}{y} + \dfrac{y}{x}\right) \cdot xy}{\left(\dfrac{1}{x}\right) \cdot xy}$$

$$= \frac{\left(\dfrac{x}{y}\right) \cdot xy + \left(\dfrac{y}{x}\right) \cdot xy}{\left(\dfrac{1}{x}\right) \cdot xy}$$

$$= \frac{x^2 + y^2}{y}.$$

(Where have we used the distributive law?)

4. $$\frac{\dfrac{1}{a} + \dfrac{1}{b}}{\dfrac{1}{a^2} + \dfrac{1}{b^2}} = \frac{\left(\dfrac{1}{a} + \dfrac{1}{b}\right) \cdot a^2b^2}{\left(\dfrac{1}{a^2} + \dfrac{1}{b^2}\right) \cdot a^2b^2}$$

$$= \frac{\left(\dfrac{1}{a}\right) \cdot a^2b^2 + \left(\dfrac{1}{b}\right) \cdot a^2b^2}{\left(\dfrac{1}{a^2}\right) \cdot a^2b^2 + \left(\dfrac{1}{b^2}\right) \cdot a^2b^2}$$

$$= \frac{ab^2 + a^2b}{b^2 + a^2}.$$

We call the fractions which are in the numerator or denominator of a complex fraction the *secondary fractions.* Their denominators are the *secondary denominators.* The rule for changing a complex fraction to an equal simple fraction is:

12.2.2 *Multiply the numerator and denominator of the complex fraction by the LCD of the secondary fractions.*

5. $$\frac{\dfrac{1}{x+y}+\dfrac{1}{x}}{\dfrac{1}{x+y}+\dfrac{1}{y}} = \frac{\left(\dfrac{1}{x+y}+\dfrac{1}{x}\right)\cdot xy(x+y)}{\left(\dfrac{1}{x+y}+\dfrac{1}{x}\right)\cdot xy(x+y)}$$

$$= \frac{xy+y(x+y)}{xy+x(x+y)}$$

$$= \frac{2xy+y^2}{2xy+x^2}.$$

Exercise 12.2.3

Change each of the following to a simple fraction in lowest terms. *Do the first ten in two different ways.*

1. $\dfrac{\frac{1}{2}+\frac{1}{3}}{\frac{1}{4}+\frac{1}{3}}$.

2. $\dfrac{\frac{3}{5}+\frac{1}{10}}{\frac{2}{5}+\frac{3}{10}}$.

3. $\dfrac{2+\frac{1}{3}}{3+\frac{1}{2}}$.

4. $\dfrac{\dfrac{a}{b}+1}{\dfrac{1}{a}+\dfrac{1}{b}}$.

5. $\dfrac{x+3}{1+\dfrac{x}{3}}$.

6. $\dfrac{x+3}{1+\dfrac{3}{x}}$.

7. $\dfrac{1+\dfrac{1}{x}}{1+\dfrac{1}{y}}$.

8. $\dfrac{\dfrac{1}{x}+\dfrac{1}{y}}{\dfrac{2}{x}+\dfrac{2}{y}}$.

9. $\dfrac{\dfrac{x}{x+y}+\dfrac{y}{x}}{\dfrac{y}{x+y}+\dfrac{x}{y}}$.

10. $\dfrac{\dfrac{a}{b^2}+\dfrac{b}{a^2}}{\dfrac{1}{a}+\dfrac{1}{b}}$.

11. $\dfrac{\dfrac{x}{2}+5}{\dfrac{x}{6}+5x+1}$.

12. $\dfrac{x+\dfrac{y^3}{x^2}}{1+\dfrac{y}{x}+\dfrac{y^2}{x^2}}$.

13. $\dfrac{\dfrac{x}{x+2}+\dfrac{x}{x+1}}{\dfrac{x}{(x+1)^2}}$.

14. $1+\dfrac{1}{1+\dfrac{1}{x}}$.

15. $1 + \cfrac{2}{a + \cfrac{3}{a + \cfrac{4}{a}}}.$

16. $\dfrac{\dfrac{a}{b} + \dfrac{b}{a}}{\dfrac{a^2}{b^2} + \dfrac{b^2}{a^2}} \cdot \dfrac{\dfrac{1}{a^2} + \dfrac{1}{b^2}}{\dfrac{1}{a} + \dfrac{1}{b}}.$

17. $\dfrac{\dfrac{x}{y} + \dfrac{y}{x}}{\dfrac{x^2}{y^2} + \dfrac{y^2}{x^2}} \div \dfrac{\dfrac{1}{x^2} + \dfrac{1}{y^2}}{\dfrac{1}{x} + \dfrac{1}{y}}.$

12.3 Signed Fractions

Just as we identified the natural number 3 with the positive integer $+3$, we now identify the fraction $\frac{3}{2}$ with $+\frac{3}{2}$. The fraction a/b (a and b natural numbers) may be considered as the *positive fraction* $+ a/b$. This is consistent with the law of signs for division. For $a \div b = (+a) \div (+b)$, and we should want this to be a positive number, or $\frac{a}{b} = (+a) \div (+b) = +\frac{a}{b}$. It also conforms with the agreement that numbers to the right of 0 (on our number scale) are positive.

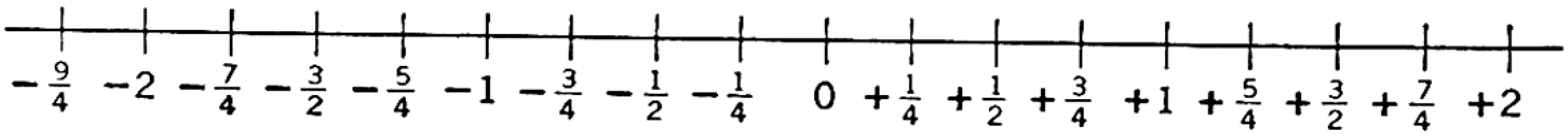


FIG. 12.3.1.

If we use our imagination, we are inspired to raise the questions: "Why restrict ourselves to positive fractions only? Why not label the fracture point half-way between -1 and 0 with the label $-\frac{1}{2}$, etc.?" We do this and invent negative fractions.

$-\frac{a}{b}$ may be considered the negative of $\frac{a}{b}$, the number which when added to $\frac{a}{b}$ gives the sum 0. Using our previous illustration of addition of signed numbers (8.4), we may think of the statement $\frac{1}{2} + (-\frac{1}{2}) = 0$ as meaning that we start at $\frac{1}{2}$ and then move $\frac{1}{2}$ unit to the left, arriving at 0.

Suppose we now remove the restriction that in the symbol a/b, a and b are natural numbers and say that a and b are signed integers ($b \neq 0$). If both a and b are positive, there is no doubt of the meaning to be given a/b. Thus $+2/+3 = \frac{2}{3}$. But, if one or both are negative, we must investigate to see what would be a good meaning to give the symbol a/b.

$(-3) \div (+2)$, if it is a number, should be a negative number (by our law of signs). We let $(-3) \div (+2) = -3/2$. Extending the statement $\frac{a}{b} = a \div b$ to include both positive and negative integers, $\frac{-3}{+2} = (-3) \div (+2)$. Thus $\frac{-3}{+2} = -\frac{3}{2}$.

Also we want $(+3) \div (-2) = -\frac{3}{2}$ if the law of signs is to hold and $\frac{+3}{-2} = (+3) \div (-2)$. Thus $\frac{+3}{-2} = -\frac{3}{2}$.

We see that, if our previous rules are to hold, then

$$\frac{-3}{+2} = \frac{+3}{-2} = -\frac{3}{2}.$$

12.3.2 *If a and b are natural numbers, then* $\frac{-a}{+b} = \frac{+a}{-b} = -\frac{a}{b}$.

Consider the quotient $(-5) \div (-4)$; if this is to be a number, we should like it to be a positive number. We let $(-5) \div (-4) = +5/4$. Also to be consistent with our previous work $(-5) \div (-4) = \frac{-5}{-4}$ and therefore

$$\frac{-5}{-4} = +\frac{5}{4} = \frac{5}{4}.$$

12.3.3 *If a and b are natural numbers, then* $\frac{-a}{-b} = \frac{+a}{+b} = +\frac{a}{b} = \frac{a}{b}$.

Note that, if we let the important rule $\frac{ax}{bx} = \frac{a}{b}$ apply to these numbers, we obtain this last result by taking $x = -1$,

$$\frac{-a}{-b} = \frac{a(-1)}{b(-1)} = \frac{a}{b}.$$

As yet we have not settled upon a meaning for $\dfrac{0}{+b}$ or $\dfrac{0}{-b}\cdot$ But if $\dfrac{0}{+b} = 0 \div (+b)$ we have already agreed that this is zero and we have

12.3.4 $$\frac{0}{+b} = 0.$$

Similarly

12.3.5 $$\frac{0}{-b} = 0 \text{ (b a natural number).}$$

12.4 The Rational Number System

The system of numbers we now have constructed consists of all numbers of the form p/q, where p and q are integers, $q \neq 0$. This system of numbers is called the rational number system.

12.4.1 DEFINITION: A *rational number* is a number which can be written in the form p/q, where p and q are integers, $q \neq 0$.

The word rational, as used in mathematics, does not mean "not insane." The important part of the word consists of the first five letters, *ratio*. The ratio of two numbers p and q means $p \div q$ and when the two numbers are integers we say that $p \div q = p/q$ is rational.

Included in the system of rational numbers are the integers themselves (since $+a = +a/1$ and $-a = -a/1$ and $0/1 = 0/q = 0$ where q is any integer except 0). The system of rational numbers may be displayed as follows:

$$\begin{array}{ccccccccccc}
\cdots, & -4, & -3, & -2, & -1, & 0, & 1, & 2, & 3, & 4, & \cdots \\
\cdots, & -\frac{4}{2}, & -\frac{3}{2}, & -\frac{2}{2}, & -\frac{1}{2}, & \frac{0}{2}, & \frac{1}{2}, & \frac{2}{2}, & \frac{3}{2}, & \frac{4}{2}, & \cdots \\
\cdots, & -\frac{4}{3}, & -\frac{3}{3}, & -\frac{2}{3}, & -\frac{1}{3}, & \frac{0}{3}, & \frac{1}{3}, & \frac{2}{3}, & \frac{3}{3}, & \frac{4}{3}, & \cdots \\
\cdots, & -\frac{4}{4}, & -\frac{3}{4}, & -\frac{2}{4}, & -\frac{1}{4}, & \frac{0}{4}, & \frac{1}{4}, & \frac{2}{4}, & \frac{3}{4}, & \frac{4}{4}, & \cdots \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot &
\end{array}$$

Here the first row contains the integers (i.e., the rational numbers with denominator 1), the second row contains the rational numbers with denominator 2, etc.

Where will the number $\frac{37}{59}$ appear in this display?
Where will the number $-\frac{427}{999}$ appear? 20

12.5 Multiplication and Division of Rational Numbers

Since $+\dfrac{a}{b} = \dfrac{+a}{+b} = \dfrac{a}{b}$, there is no need for further discussion of the multiplication of two positive rational numbers. The rules we adopted for multiplication of fractions will apply to positive rational numbers. The product of two positive rational numbers is a rational number whose numerator is the product of the numerators of the given numbers and whose denominator is the product of their denominators. *We extend this law to all rational numbers.*

EXAMPLES 12.5.1

1. $$\frac{2}{3}\cdot\left(-\frac{5}{7}\right) = \frac{2}{3}\cdot\frac{-5}{7} = \frac{-10}{21} = -\frac{10}{21}.$$

2. $$\left(-\frac{5}{9}\right)\cdot\left(-\frac{3}{7}\right) = \frac{-5}{9}\cdot\frac{-3}{7} = \frac{15}{63} = \frac{5}{21} \text{ or}$$
$$\left(-\frac{5}{9}\right)\cdot\left(-\frac{3}{7}\right) = \left(\frac{5}{-9}\right)\left(\frac{3}{-7}\right) = \frac{15}{63} = \frac{5}{21} \text{ or}$$
$$\left(-\frac{5}{9}\right)\cdot\left(-\frac{3}{7}\right) = \left(\frac{-5}{9}\right)\left(\frac{3}{-7}\right) = \frac{-15}{-63} = \frac{15}{63} = \frac{5}{21}.$$

3. $$\left(\frac{a}{b}\right)\left(-\frac{b}{a^2}\right) = \left(\frac{a}{b}\right)\left(\frac{-b}{a^2}\right) = \frac{-ab}{a^2b} = \frac{-1(ab)}{a(ab)} = \frac{-1}{a} = -\frac{1}{a}.$$

4. $$\frac{x-y}{x+y}\cdot\frac{x}{y-x} = \frac{x-y}{x+y}\cdot\frac{-x}{x-y} = \frac{-x}{x+y} = -\frac{x}{x+y}.$$

Similarly division of rational numbers follows the same rules as division of fractions.

5. $$\left(-\frac{2}{3}\right)\div\left(\frac{4}{5}\right) = \left(\frac{-2}{3}\right)\div\left(\frac{4}{5}\right) = \frac{-2}{3}\cdot\frac{5}{4} = -\frac{5}{6}.$$

Since $\dfrac{a}{b}\cdot\dfrac{c}{d} = \dfrac{ac}{bd}$ and, if a, b, c, d are integers, then ac and bd are integers, i.e., $\dfrac{ac}{bd}$ is a rational number, we see that *rational numbers are closed under the operation of multiplication.*

Similarly, $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}$ and *rational numbers are closed under the operation of division.*

EXERCISE 12.5.2

Perform the indicated operations and reduce answers to lowest terms.

1. $\frac{3}{5} \cdot \frac{10}{9}$.
2. $(-\frac{5}{8})(\frac{24}{15})$.
3. $(-\frac{4}{9})(-\frac{18}{24})$.
4. $-\frac{4x}{5y} \cdot \frac{15y}{8x}$.
5. $\left(-\frac{3x}{2y}\right)\left(-\frac{8xy}{9}\right)$.
6. $(-5a)\left(\frac{4a}{15b}\right)$.
7. $-\frac{a-b}{a+b} \cdot \frac{3a+3b}{2a-2b}$.
8. $\frac{b-a}{a+b} \cdot \frac{3a+3b}{2a-2b}$.
9. $-\frac{1}{3} \div \frac{5}{3}$.
10. $\frac{3}{7} \div (-\frac{5}{3})$.
11. $\frac{9}{8} \div (-\frac{21}{20})$.
12. $-3 \div \frac{6}{7}$.
13. $-9 \div (-\frac{21}{8})$.
14. $-\frac{x}{y} \div \frac{y}{x}$.
15. $\frac{3x}{2y} \div \left(-\frac{x}{y}\right)$.
16. $-\frac{x^2}{y} \div \frac{3x^3}{y^2}$.
17. $\left(-\frac{x}{y}\right)^2 \div \left(-\frac{x^2}{y^2}\right)$.
18. $\frac{p^3}{(2p)^3} \div \frac{(-p)^3}{-p^3}$.
19. $\frac{(-a)^4}{(-3b)^3} \div \frac{-2a^3}{(3b)^2}$.
20. $\frac{(2rs^2)^3}{(3p^2q)^2} \div \frac{-r^2s}{(6pq^2)^2}$.
21. $\frac{x^2y - xy^2}{(xy)^2 + y^2} \div \frac{2xy^2 - 2x^2y}{3x^2 + 3}$.
22. $-\frac{3a^2 + 3a}{(ab^2 + b^2)^2} \cdot \frac{3b(ab + b)^2}{6a - 3b}$.
23. $\frac{3ax^2 - 9ax}{10x^2 + 5x} \div \frac{3a^2 - a^2x}{2x^3 + x^2}$.
24. $-\frac{2a}{(-b)^2} \div \left(\frac{c}{d^2} \cdot \frac{6c^2}{b(-d^2)}\right)$.

12.6 Addition and Subtraction of Rational Numbers

Addition of rational numbers shall follow the same rules as for addition of fractions. Subtraction of rational numbers is, as with integers, equivalent to adding the negative of the subtrahend.

EXAMPLES 12.6.1

1. $\frac{2}{3} + \frac{-4}{3} = \frac{2 + (-4)}{3} = \frac{2 - 4}{3} = \frac{-2}{3} = -\frac{2}{3}.$

2. $\frac{3}{5} + \left(-\frac{4}{5}\right) = \frac{3}{5} + \frac{-4}{5} = \frac{3 - 4}{5} = -\frac{1}{5}.$

3. $-\frac{6}{7} + \left(-\frac{9}{7}\right) = \frac{-6 - 9}{7} = -\frac{15}{7}.$

4. $\frac{5}{8} - \frac{1}{8} = \frac{5}{8} + \left(-\frac{1}{8}\right) = \frac{5 - 1}{8} = \frac{4}{8} = \frac{1}{2}.$

5. $\frac{2}{3} + \frac{-1}{4} = \frac{8}{12} + \frac{-3}{12} = \frac{8 - 3}{12} = \frac{5}{12}.$

6. $-\frac{5}{6} - \frac{7}{8} = -\frac{20}{24} - \frac{21}{24} = \frac{-20 - 21}{24} = -\frac{41}{24}.$

7. $\frac{a}{x} - \frac{b}{x} = \frac{a}{x} + \frac{-b}{x} = \frac{a + (-b)}{x} = \frac{a - b}{x}.$

8. If a, b, c, d are integers, then $\frac{a}{b} + \frac{c}{d} = \frac{ad}{bd} + \frac{bc}{bd} = \frac{ad + bc}{bd}.$

9. If a, b, c, d are integers, then $\frac{a}{b} - \frac{c}{d} = \frac{ad}{bd} - \frac{bc}{bd} = \frac{ad - bc}{bd}.$

From Examples 8 and 9, respectively, we conclude that *rational numbers are closed under the operations of addition and subtraction.*

Since rational numbers are closed under the operations of addition, multiplication, subtraction, and division these four operations are called the *rational operations.*

EXERCISE 12.6.2

Perform the indicated operations and reduce answers to lowest terms:

1. $\frac{7}{16} - \frac{1}{16}$.
2. $-\frac{19}{24} + \frac{23}{24}$.
3. $-\frac{11}{5} - \frac{14}{5}$.
4. $\frac{2}{3} - \frac{5}{2}$.
5. $\frac{4}{5} - \frac{1}{20}$.
6. $\frac{2}{5} - 2$.
7. $-\frac{25}{16} - \frac{18}{32}$.
8. $\frac{4}{a} - \frac{6}{a}$.
9. $\frac{3}{x} - \frac{2a}{x}$.
10. $-\frac{m}{3} - \frac{m}{2}$.
11. $\frac{r}{6} - \frac{s}{8}$.
12. $\frac{x}{2} + \frac{x}{6} - \frac{x}{4}$.
13. $\frac{7t}{2} - \frac{t}{6} - \frac{3t}{4}$.
14. $\frac{r-2}{3} - \frac{r+2}{6}$.
15. $\frac{3a-b}{4} - \frac{2a-3b}{10}$.
16. $\frac{3}{x} - \frac{2}{y}$.
17. $-\frac{2}{a} + \frac{5}{b}$.
18. $-\frac{3}{c} - \frac{4}{d}$.
19. $\frac{3}{x} - \frac{2}{x^2}$.
20. $\frac{1}{3x} - \frac{2}{x} + \frac{3}{4x}$.
21. $\frac{1}{ab} + \frac{2}{bc} - \frac{3}{ac}$.
22. $\frac{5}{pq} - \frac{7}{qr}$.
23. $\frac{1}{x^2} - \frac{1}{xy} + \frac{3}{y^2}$.
24. $\frac{3}{x^3} - \frac{2}{x^2} + \frac{3}{x}$.
25. $-\frac{5}{x^2y} + \frac{4}{xy^2}$.
26. $\frac{a+1}{2a} - \frac{3a+2}{4a}$.
27. $\frac{m+n}{2m} - \frac{m-n}{3n}$.
28. $\frac{4}{3p^2} - \frac{3}{4pq} + \frac{2}{5q^2}$.
29. $\frac{5}{2x^2} + \frac{1}{6x} - \frac{2}{3}$.
30. $\frac{5a}{2b^3} - \frac{4b}{3a^3} + \frac{2}{5b} - \frac{5}{6a^2b^2}$.
31. $\frac{1}{x+1} - \frac{1}{2x+2}$.
32. $\frac{1}{2x-2y} + \frac{2}{3x-3y}$.

33. $\dfrac{7}{8a + 8b} - \dfrac{3}{5a + 5b} - \dfrac{5}{4a + 4b}.$

34. $\dfrac{1}{x - 1} - \dfrac{1}{x + 1}.$

35. $-\dfrac{2}{p - q} + \dfrac{3}{p + q}.$

36. $\dfrac{x + 3}{x} - \dfrac{x}{x + 3}.$

37. $\dfrac{a + 1}{a + 2} - \dfrac{a + 2}{a + 3}.$

38. $\dfrac{u - v}{u + v} + \dfrac{u + v}{u - v}.$

39. $2 - \dfrac{3}{a}.$

40. $b - 2 - \dfrac{3}{2b}.$

41. $x + \dfrac{2}{3x} - 1.$

42. $3 + \dfrac{2}{x + 1}.$

43. $-2 + \dfrac{5}{2y - 1}.$

44. $a - \dfrac{b^2}{a + b}.$

45. $3 - \dfrac{2x - 6y}{x - 2y}.$

46. $p - \dfrac{p^2 - 1}{p}.$

47. $2 - \dfrac{1}{x - 1} + \dfrac{1}{x + 1}.$

48. $5 + \dfrac{x}{x + 1} - \dfrac{x}{x - 1}.$

49. $2x - \dfrac{x^2 - xy}{x - y}.$

50. $a - b + \dfrac{2b^2}{a + b}.$

51. $\left(\dfrac{x}{y} - 1\right)\left(\dfrac{x}{y} + 1\right).$

52. $\left(\dfrac{a^2}{b} - b\right) \div \left(\dfrac{a}{b} + 1\right).$

53. $\left(\dfrac{p}{q} + \dfrac{q}{p}\right) \div \left(\dfrac{p}{q} - \dfrac{q}{p}\right).$

54. $\left(\dfrac{r}{2s} - \dfrac{2s}{r}\right) \div \left(\dfrac{r}{s} - \dfrac{4s}{r}\right).$

55. $\left(a - \dfrac{1}{a + 1}\right) \div \left(a - \dfrac{2a - 1}{a + 1}\right).$

56. $\dfrac{x - y}{y - x}.$

57. $\dfrac{3}{x - y} + \dfrac{2}{y - x}.$

58. $\dfrac{p}{p - q} - \dfrac{q}{q - p}.$

59. $\dfrac{-c - d}{-c} + \dfrac{c - d}{c}.$

60. $\dfrac{3a}{a - b} + \dfrac{2b}{3b - 3a}.$

Remove symbols of grouping, combine like terms, and place over a common denominator:

61. $\left(\frac{3}{5} + \frac{2}{7}\right) - \left(\frac{1}{3} - \frac{4}{5}\right).$

62. $\frac{2}{15} - 3\left(\frac{1}{5} + \frac{3}{10}\right).$

63. $\frac{x}{2} - 3\left(\frac{x}{4} + \frac{1}{9}\right).$

64. $x^2 - \left(\frac{1}{2} - x + \frac{x^2}{3}\right).$

65. $\frac{x}{2}\left(x^2 - \frac{3x}{2} - \frac{1}{3}\right) - \left(x^2 + \frac{3x}{4} - \frac{1}{6}\right).$

66. $2x^3\left(\frac{x^2}{3} - \frac{x}{2} + \frac{1}{12}\right) - \frac{1}{4}(x^5 - 2x^4 + x^2 + 5).$

67. $\frac{-3x}{2}(x^3 + 2x^2 - 4x) - x\left[2x^2 - \frac{3x}{2}(x^2 + 1)\right].$

68. $-x^3\left\{x^2 - \left[\frac{2x}{3}(x^2 - 2x + 1) + \frac{3}{2}(x - 1)\right]\right\}.$

Perform the indicated operations and reduce to lowest terms:

69. $\left(\frac{x}{y}\right)^2\left(\frac{x}{y}\right)^3.$

70. $\frac{1}{x^2} \cdot x^5.$

71. $a^2 \cdot \frac{1}{a^6}.$

72. $p^7 \cdot \frac{1}{p^7}.$

73. $\left(\frac{1}{x}\right)^2 \cdot \left(\frac{x}{y}\right)^4.$

74. $\left(\frac{-a}{b^2}\right)^3 \cdot \left(\frac{b^3}{a^2}\right)^3.$

75. $\left(\frac{-a}{b^2}\right)^2 \cdot \left(\frac{b}{a}\right)^2.$

76. $\frac{2s^2}{3t^3} \cdot \left(\frac{-3t^2}{4s}\right)^3.$

77. $\frac{(xy)^2}{x^2y^2} \cdot \frac{(xy^2)^2}{(x^2y)^2}.$

78. $\frac{p^2}{q^3} \cdot \frac{p^3}{q^2} \cdot (p^2q^3)^2.$

12.7 "Greater Than" and "Less Than" for Rational Numbers

Definition 8.9.1 stated that $a > b$ if $a - b$ is positive, $a < b$ if $a - b$ is negative, and $a = b$ if $a - b = 0$. At the time at which we formulated this definition we were considering signed integers. However, this same definition applies to all numbers. As before, the greater number lies to the right of the smaller number on our number scale.

EXERCISE 12.7.1

Determine which number is the greater:

1. $\frac{2}{3}, \frac{7}{9}$.
2. $\frac{19}{24}, \frac{29}{36}$.
3. $\frac{85}{48}, \frac{71}{40}$.
4. $\frac{59}{160}, \frac{149}{400}$.
5. $\frac{31}{71}, 0$.
6. $-\frac{15}{32}, 0$.
7. $-19, \dfrac{1}{1{,}000{,}000}$.
8. $\dfrac{1}{1{,}000{,}000}, -999$.
9. $0, -728$.
10. $-\dfrac{19}{24}, \dfrac{-29}{36}$.
11. $\dfrac{-68}{175}, -\dfrac{97}{250}$.
12. $\frac{170}{189}, \frac{1699}{1889}$.
13. $-\dfrac{301}{180}, \dfrac{136}{-81}$.
14. $\dfrac{-144}{245}, \dfrac{431}{-735}$.

12.8 Some More Laws of Exponents

In doing Problems 69 through 78 of Exercise 12.6.2 you probably noticed that there seem to be some additional laws of exponents which apply to these problems. For instance, in doing Problem 69 we have $\left(\dfrac{x}{y}\right)^2$ which means $\left(\dfrac{x}{y}\right) \cdot \left(\dfrac{x}{y}\right)$, and by the rule for multiplying rational numbers this is $\dfrac{x \cdot x}{y \cdot y} = \dfrac{x^2}{y^2}$. This is an example of the general rule:

12.8.1 *If n is a positive integer, then*

$$\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}.$$

In Problem 70 we have $\dfrac{1}{x^2} \cdot x^5 = \dfrac{1}{x^2} \cdot \dfrac{x^5}{1} = \dfrac{x^5}{x^2} = \dfrac{x^3 \cdot x^2}{1 \cdot x^2} = \dfrac{x^3}{1} = x^3$ and in Problem 71 we have $a^2 \cdot \dfrac{1}{a^6} = \dfrac{a^2}{a^6} = \dfrac{1 \cdot a^2}{a^4 \cdot a^2} = \dfrac{1}{a^4}$. These are examples of the general rule:

12.8.2 *If $x \neq 0$ and m and n are positive integers, then*

$$\frac{x^m}{x^n} = x^{m-n} \qquad \text{if } m > n,$$

$$\frac{x^m}{x^n} = \frac{1}{x^{n-m}} \qquad \text{if } m < n,$$

$$\frac{x^m}{x^n} = 1 \qquad \text{if } m = n.$$

EXAMPLES 12.8.3

1. $\left(\frac{a}{b}\right)^3 = \frac{a}{b} \cdot \frac{a}{b} \cdot \frac{a}{b} = \frac{a \cdot a \cdot a}{b \cdot b \cdot b} = \frac{a^3}{b^3}.$
2. $\left(\frac{-2}{3x}\right)^4 = \frac{-2}{3x} \cdot \frac{-2}{3x} \cdot \frac{-2}{3x} \cdot \frac{-2}{3x} = \frac{(-2)^4}{(3x)^4} = \frac{16}{81x^4}.$
3. $\frac{a^5}{a^2} = \frac{a \cdot a \cdot a \cdot a \cdot a}{a \cdot a} = \frac{a^3 \cdot a^2}{1 \cdot a^2} = \frac{a^3}{1} = a^3$, if $a \neq 0$.
4. $\frac{s^5}{s^7} = \frac{1}{s^2}$, if $s \neq 0$.
5. $\frac{2t^4}{3t^2} = \frac{2t^2}{3}$, if $t \neq 0$.
6. $\frac{(2r)^4}{2r^6} = \frac{2^4 r^4}{2r^6} = \frac{2^3}{r^2}$, if $r \neq 0$.
7. $\frac{x^7}{x^7} = 1$, if $x \neq 0$.

EXERCISE 12.8.4

Apply the laws of exponents and reduce to lowest terms:

1. $\left(\frac{x}{y}\right)^5.$
2. $\left(-\frac{3}{2a}\right)^3.$
3. $\frac{a^t}{b^{2t}}.$
4. $\frac{(-2x)^3}{2^5 x}.$
5. $\frac{(x^5 y)(xy^5)}{(xy)^6}.$
6. $\frac{-p^3(-q)^3}{-p^2(-q)^2}.$

7. $\dfrac{(ab^2)^3}{(a^3b)^2}$.

8. $\dfrac{(6a^5b^4)^3}{(2a^2b^3)^2(3ab)^3}$.

9. $\dfrac{(-2ax)^3(-3a^2x)^2}{(12ax^2)^2(-a^2x^2)^3}$.

10. $\left(\dfrac{x^2}{y}\right)^2 \cdot \left(\dfrac{x}{y^2}\right)^2$.

11. $\left(\dfrac{p^2q}{r}\right)^4 \left(\dfrac{p}{q^2r^3}\right)\left(\dfrac{r^2}{p^2}\right)^5$.

12. $\dfrac{x-y}{(x-y)^2}$.

13. $\dfrac{(x+y)^2}{(x^2-y^2)^2}$.

14. $\dfrac{a^x}{a^{x-1}}$.

15. $\dfrac{r^{p+2}}{r^{2p+3}}$.

16. $\dfrac{(a^{x-1})(a^{x+1})}{a^{3x}}$.

17. $\dfrac{(2^{m-n})2^n}{2^{m+n}}$.

18. $\dfrac{(2^m)^n}{(2^n)^m}$.

19. $\left(\dfrac{a}{b}\right)^s \div \dfrac{a^{2s}}{b^{3s}}$.

20. $\dfrac{x^{s^2}}{x^s}$.

21. $\dfrac{(x^s)^2}{x^s}$.

22. $\left(\dfrac{a^2}{-2b}\right)^3 \div \left(\dfrac{-a}{4b}\right)^2$.

23. $\dfrac{(-2a^mb^n)^3}{(a^{2m}b^n)^2}$.

24. $\dfrac{(x^ay)^c}{(x^{a-1})^c}$.

Perform the indicated operations and reduce answers to lowest terms. Evaluate the results for the given values of the letters; check by substituting the given values in the original expression and then performing the operations.

25. $\dfrac{\left(\dfrac{x}{y}\right)^2 - \left(\dfrac{y}{x}\right)^2}{x^2+y^2}$; $x = 2$, $y = 3$.

26. $\dfrac{\left(\dfrac{1}{x}\right)^3 + \left(\dfrac{1}{y}\right)^3}{\left(\dfrac{x}{y}\right)^4}$; $x = \dfrac{1}{2}$, $y = \dfrac{1}{3}$.

27. $\dfrac{a^2+b^2}{\left(\dfrac{1}{ab}\right)^2}$; $a = b = \dfrac{2}{3}$.

28. $\dfrac{(p^3)^T}{(p^T)^2}$; $p = -2,\ T = 2.$

29. $\dfrac{x^2 + \dfrac{1}{x^2}}{x^3 + \dfrac{1}{x}}$; $x = -\dfrac{1}{2}.$

30. $\dfrac{(x^2)^a + \dfrac{1}{(x^a)^2}}{(x^a)^3 + \dfrac{1}{x^a}}$; $x = -\dfrac{2}{3},\ a = 1.$

31. $\dfrac{x + 2 - \dfrac{3}{x}}{1 + \dfrac{2}{x} - \dfrac{3}{x^2}}$; $x = -\dfrac{2}{3}.$

32. $\dfrac{2x^2 + 3x - 5 + \dfrac{1}{x}}{7 - \dfrac{2}{x} + \dfrac{5}{x^2}}$; $x = -4.$

33. $\dfrac{2x^2 + 3x - 5 + \dfrac{1}{x - 1}}{7 - \dfrac{2}{x - 1} + \dfrac{5}{(x - 1)^2}}$; $x = 2.$

13

Special Products; Polynomials

13.1 A Brief Summary

At this point it would be well for us to review briefly the progress we have made in learning to play our "game." The equipment we now have is the set of rational numbers, all numbers which can be expressed in the form p/q, where p and q are integers. This set of numbers is closed under the four fundamental operations of addition, multiplication, subtraction, and division. Division by zero is never permitted. Also members of this set of numbers obey the following rules. (Each letter stands for a rational number.)

The commutative law for addition:

$$a + b = b + a.$$

The associative law for addition:

$$(a + b) + c = a + (b + c).$$

The commutative law for multiplication:

$$ab = ba.$$

The associative law for multiplication:

$$(ab)c = a(bc).$$

The distributive law:

$$a(b + c) = ab + ac$$

and more generally

$$a(p + q + \cdots + z) = ap + aq + \cdots + az.$$

The laws of exponents (m and n natural numbers or positive integers):

$$x^m x^n = x^{m+n},$$

$$(x^m)^n = x^{mn},$$

$$(xy)^m = x^m y^m,$$

$$\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n},$$

$$\frac{x^m}{x^n} = x^{m-n} \qquad \text{if } m > n,\ x \neq 0,$$

$$\frac{x^m}{x^n} = \frac{1}{x^{n-m}} \qquad \text{if } m < n,\ x \neq 0,$$

$$\frac{x^m}{x^n} = 1 \qquad \text{if } m = n,\ x \neq 0.$$

The law of signs for multiplication: the product of two numbers of like signs is positive, the product of two numbers of unlike signs is negative, and the product of any number and zero is zero.

We made the agreement that, in an expression involving some or all of the four fundamental operations, multiplications and divisions are to precede additions and subtractions except where parentheses indicate otherwise.

13.2 Special Products

As you progress with your study of algebra you will find certain computations used so often that it will save time and energy if you learn to perform them rapidly, just as it saves time and energy to learn the multiplication tables in arithmetic.

We shall now discover some special rules which will enable us to obtain certain products by inspection. We shall then discover how to determine the factors of certain special products. Just as we learn that 36 can be factored into 4 times 9 by first learning that 4 times 9 is 36, we learn to factor algebraic expressions by first becoming adept at multiplication. Ability in factoring comes from experience with multiplication.

It is advisable to become familiar at once with the following types of products.

13.2.1
1. $a(b + c - d + e - f) = ab + ac - ad + ae - af.$
2. $(a + b)(a - b) = a^2 - b^2.$
3. $(a + b)^2 = a^2 + 2ab + b^2.$
4. $(a - b)^2 = a^2 - 2ab + b^2.$
5. $(x + b)(x + d) = x^2 + (b + d)x + bd.$
6. $(ax + b)(cx + d) = acx^2 + (ad + bc)x + bd.$

Each of these can be verified by multiplication (using the distributive law). Indeed, type 1 is simply the (generalized) distributive law.

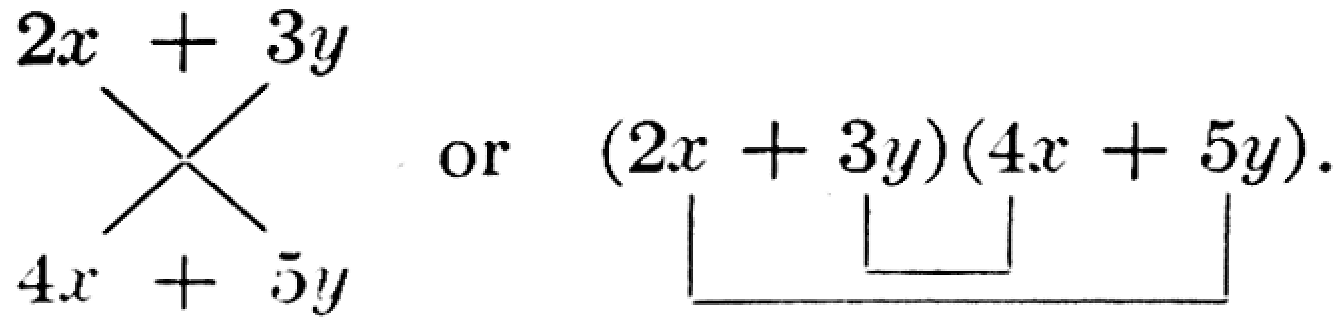
Types 2, 3, and 4 should be learned in words. For instance, type 2 states, "the product of the sum and difference of two numbers equals the square of the first number minus the square of the second."

State types 3 and 4 in words.[21]

In using types 5 and 6 we should mentally use the distributive law. Thus

$$\begin{aligned}(x + 3)(x + 2) &= (x + 3)x + (x + 3)2\\ &= x^2 + 3x + 2x + 6\\ &= x^2 + 5x + 6.\\ (x - 4)(x + 3) &= x^2 - 4x + 3x - 12 = x^2 - x - 12.\\ (2x + 3y)(4x + 5y) &= 8x^2 + 12xy + 10xy + 15y^2\\ &= 8x^2 + 22xy + 15y^2.\\ (x - 3)(2x + 7) &= 2x^2 - 6x + 7x - 21\\ &= 2x^2 + x - 21.\end{aligned}$$

The "middle term" of each of these products is found by finding the sum of the "cross products" of the terms of the factors:



EXERCISE 13.2.2

Multiply out by inspection:

1. $3x^2(x^2 - 2xy + 3y^2)$.
2. $-2p^2q^2(5p^2 - 7pq + 11q^2)$.
3. $2^3a^2b(2^5a^3 + 2^4a^2b - 2^3ab^2 + 2^2ab^3)$.

4. $4^3xy(2^2x^2 - 2xy + y^2)$.
5. $(x + y)(x - y)$.
6. $(3p + q)(3p - q)$.
7. $(2r + 3s)(2r - 3s)$.
8. $(x^2 + y)(x^2 - y)$.
9. $(a^2 + b^2)(a^2 - b^2)$.
10. $(2x - 3y)(2x + 3y)$.
11. $(3pq - 1)(3pq + 1)$.
12. $(4ab - 5cd)(4ab + 5cd)$.
13. $(x^2 - y^3)(x^2 + y^3)$.
14. $(r^2s - rs^2)(r^2s + rs^2)$.
15. $(3a^3 - 2b^3)(3a^3 + 2b^3)$.
16. $(x + 1)^2$.
17. $(x - 1)^2$.
18. $(x + 3)^2$.
19. $(x - 3)^2$.
20. $(2p + q)^2$.
21. $(2p - q)^2$.
22. $(3s - 2t)^2$.
23. $(3s + 2t)^2$.
24. $(ab + 3)^2$.
25. $(1 + 2ab)^2$.
26. $(1 - 2ab)^2$.
27. $(x^2 + y)^2$.
28. $(x + y^2)^2$.
29. $(2u - 3v^2)^2$.
30. $(3u + 2v^2)^2$.
31. $(4p^2 - 3q^2)^2$.
32. $(3p^2 + 4q^2)^2$.
33. $(2ab^2 - 3a^2b)^2$.
34. $(3a^2b + 2ab^2)^2$.
35. $(x + 2)(x - 1)$.
36. $(x - 2)(x - 1)$.
37. $(a - 7)(a + 2)$.
38. $(a + 11)(a - 3)$.
39. $(p - 5)(p - 3)$.
40. $(p + 7)(p - 4)$.
41. $(2r + 1)(r + 2)$.
42. $(2r + s)(r + 2s)$.
43. $(2b - 5c)(3b - 5c)$.
44. $(3b - 7c)(2b - 6c)$.
45. $(x^2 + 2y)(x^2 + 3y)$.
46. $(2x^2 - 3y)(x^2 + 2y)$.
47. $(u^2 + v^2)(2u^2 + 3v^2)$.
48. $(u^2 + v^2)(2u^2 - 3v^2)$.
49. $[x + (y + z)][x - (y + z)]$.
50. $(x + y + z)(x - y - z)$.
51. $[(a + b) + c][(a + b) - c]$.
52. $(a + b + c)(a + b - c)$.
53. $[(p + q) + r]^2$.
54. $[p + (q + r)]^2$.
55. $(p + q + r)^2$.
56. $(p + q - r)^2$.
57. $(p - q + r)^2$.
58. $(p - q - r)^2$.
59. $[2x + (3 - 2y)]^2$.
60. $[(2x + 3) - 2y]^2$.
61. $(3x + y + 2z)^2$.
62. $(4x - 2y + 3z)^2$.
63. $(a + b - 2c)(a + b + 2c)$.

64. $(a - b - 2c)(a + b + 2c)$.
65. $[(a + b) + (c + d)][(a + b) - (c + d)]$.

66. $(a + b + c + d)(a + b - c - d)$.
67. $(a + b - c + d)(a + b + c - d)$.
68. $(x - y)(x + y)(x^2 + y^2)$.
69. $(2x - 3)(4x^2 + 9)(2x + 3)$.
70. $(x^2 + xy + y^2)(x^2 - xy - y^2)$.
71. $(x^2 + xy - y^2)(x^2 + xy + y^2)$.
72. $(2x^2 - 3x + 1)(2x^2 + 3x - 1)$.

13.3 Constants and Variables

A constant is a symbol which stands for one particular number throughout a discussion. Of course, specific numbers such as 1, $\frac{1}{2}$, -3, 0, $1\frac{3}{5}$, $-1\frac{1}{2}$, π are constants. Each stands for a particular number throughout *all* discussions. But we also may have *literal constants*. These are generally symbolized by the early letters of the alphabet, such as a, b, c, d. A literal constant stands for one quantity throughout a discussion, but it may stand for different quantities as we change from one discussion to another. Sometimes, instead of using different letters to stand for different literal constants, we use just one letter with *subscripts* such as a_0, a_1, a_{17}. The numbers, usually positive integers or zero, written to the right of and below the letter are the subscripts. They are merely tags used to distinguish one symbol from the others. They have nothing to do with the size of the constant under discussion; it is possible for a_1 to be greater than a_{17}.

Contrasted with constants we have *variables*. These are symbols which may take on different values throughout a discussion. They are generally symbolized by the later letters of the alphabet, such as x, y, z, u, s, t.

Example 13.3.1

If a stone is thrown straight upward with an initial velocity of v_0 feet per second (that is, it leaves the thrower's hand at that velocity), after t seconds it will be $v_0t - \frac{1}{2}gt^2$ feet above the thrower. We may state this in the form of a formula. If we let s stand for "feet above the thrower," we have

$$s = v_0t - \tfrac{1}{2}gt^2.$$

Here v_0, $\frac{1}{2}$, 2, and g are constants. v_0 and g are literal constants. g is slightly different at different points on the earth's surface. Its value depends upon the distance the thrower is from the center of the earth; the closer he is to the center the greater is g. Since the earth is an oblate spheroid, that is, in the shape of a ball that has been stepped on, g is greater at the North Pole than it is at the equator. But in a specific problem we take a specific value for g. Also v_0 will vary from problem to problem. Even if we remain at one particular spot on the earth's surface, we all know that the distance a thrown stone will rise into the air depends upon how hard we throw it.

s and t are variables. They may take on various values in any given problem. For instance, if $g = 32$ and the stone leaves the thrower's hand at 120 feet per second, then for this discussion $v_0 = 120$ and $s = 120t - 16t^2$. We may now ask, How high is the stone at the end of 1, 2, 3, 4, $\frac{3}{2}$, $\frac{2}{3}$ seconds? What are the answers to these questions? [22] How many more questions like this can we ask? [23] How many values may s and t take on in this discussion? [24]

13.4 Rational Algebraic Expressions; Polynomials

By a *rational algebraic expression* we mean an expression which can be built up from a number of constants and variables connected by symbols of rational operations: addition, multiplication, subtraction, and division.

Examples 13.4.1

The following are each rational algebraic expressions:

1. $\dfrac{3x^2 - 2xy + y^2}{4x^3 - 5y^3}$. We say this is a rational algebraic expression in x and y (the variables).
2. $2x^3 + 4ax - b$ is a rational algebraic expression in x.
3. $3x^5$ is a rational algebraic expression in x.
4. $\dfrac{a}{x + y + z} - \dfrac{b}{x - y + z}$ is a rational algebraic expression in x, y, and z.

A rational algebraic expression which can be written without any *variables* appearing in the *divisor* is said to be an *integral* rational algebraic expression. Expressions 2 and 3 above are integral rational algebraic expressions. Instead of this mouth-filling term, integral rational algebraic expression, we usually call such an expression a *polynomial.*

5. $3x^3 + 2x^2 - 5$ is a polynomial in x.
6. $a_0x^3y + a_1x^2y^2 + a_2xy^3$ is a polynomial in x and y.
7. x^2 is a polynomial in x.
8. $15x^4 + 4x^3 - 2x^2 + 7x - 3$ is a polynomial in x.
9. $\dfrac{x^3 - 3x^2 + 2}{5}$ is a polynomial in x.
10. $\frac{1}{3}x^5 - \frac{3}{7}x^2 + 5$ is a polynomial in x.
11. $x^2 + \dfrac{1}{x}$ is a rational algebraic expression in x. It is not a polynomial.

The parts of a polynomial connected by + or − signs (if any) are called the *terms* of the polynomial. Expressions 2, 5, and 6 each have three terms. Expressions 3 and 7 each have one term; expression 8 has five terms.

A polynomial with just one term is called a monomial; expression 3 above is a monomial. A polynomial with just two terms is called a binomial; a polynomial with just three terms is called a trinomial.

12. $5x + 3$ is a binomial.
13. $7x^2 - 3x + 11$ is a trinomial.

The degree of a monomial in one variable is the exponent of that variable: $3x^2$ is of degree two; $5x^7$ is of degree seven.

The degree of a monomial in more than one variable is the sum of the exponents of all variables: $4x^2y^3$ is of degree five; $3x^3y$ is of degree four; $11x^2y^3z$ is of degree six.

We also say that $4x^2y^3$ is of degree two *in x* and of degree three *in y.*

The degree of a polynomial is the degree of its term of highest degree whose coefficient is not zero.

$$4x^3 + 6x^2 + 2x^5 - 3$$

is of degree 5;

$$1 + 3x + 4x^2 - 7x^3 + 0x^4$$

is of degree 3. A polynomial of degree 2 is called a *quadratic.* Thus expression 13 of Examples 13.4.1 is a quadratic trinomial.

A polynomial in one variable is said to be *in standard form* when all parentheses enclosing variables have been removed, like terms have been combined, and terms have been arranged in the order, from left to right, of descending powers of the variable.

$$3(x^2 + 5x - 1) - 2x(x^2 - 3) + 5x^4$$

written in standard form would be

$$5x^4 - 2x^3 + 3x^2 + 21x - 3.$$

Exercise 13.4.2

Write in standard form and state the degree of the polynomial. Show that the value of the polynomial for the given value of x is the same whether the polynomial is in the given form or in standard form.

1. $4x^2 - 2x + 5 - 7x + x^2;\ x = 2.$
2. $3(x^3 + 2x - 7) - 2(x - 4);\ x = 2.$
3. $(7x^3 - 2x^2 + 3x - 5) - (4x^3 - 2x^2 - 3x + 5);\ x = -3.$
4. $-2(x^3 + 5x^2 + 2x - 1) - (3 - 2x + 4x^2 - 2x^3);\ x = -3.$
5. $3[x + x^2(x - 2)] + 2[5x^2 - 2(x^2 - x - 1)];\ x = 5.$
6. $(x + 1)(x - 1) + (x - 1)(x + 2);\ x = -1.$
7. $(x - 2)^2 + (x + 2)^2 + (x - 2)(x + 2);\ x = -2.$
8. $(2x + 1)^2 - (2x - 1)^2 + (2x + 3)(2x - 1);\ x = -\frac{3}{2}.$
9. $(2x + 3)^2 - (2x - 3)^2 - (2x + 3)(2x - 3);\ x = -\frac{3}{2}.$
10. $x(x - 3)^2 - x^2(x + 3)^2 + 3(x - 3)(x + 3);\ x = \frac{2}{3}.$
11. $(x - 1)(x + 1)(x^2 + 1) - (x^2 + 2)^2;\ x = \frac{2}{3}.$
12. $2[x(x - 4)(x + 4) - (x - 3)^2] + (x + 2)^2;\ x = -\frac{2}{3}.$
13. $5\{x - 3x^2[x^2 - (x + 2)(x - 3)] + x^2(2x + 1)(3x - 2)\};\ x = \frac{3}{5}.$
14. $2(x^2 - 2)(x^2 + 2) - x[x^2 + (x - 2)(x + 2)];\ x = -\frac{3}{5}.$

State the degree of the polynomial and its degree in x and in y:

15. $x^2 + y^2.$
16. $x^2y^2.$
17. $x^2y + xy^2.$
18. $3x^2 + 2x^4y - xy^2 + y^3.$
19. $3x(x^2 + y^2) - 2xy(x^2 - y^2).$
20. $2x^2[x(x - y) + 4y(x + y)] - y^2[3y(y - x) - 5x(y + 2x)].$

14

Factoring

14.1 What Is Factoring?

Factoring an expression is the process of finding two or more expressions whose product is the given expression. This statement is ambiguous unless we state what *kind* of factors we want. For instance, the number 12 can be written as the product of two or more numbers in many ways:

$$12 = 6 \cdot 2,$$

$$12 = 4 \cdot 3,$$

$$12 = 2^2 \cdot 3,$$

$$12 = \tfrac{1}{5} \cdot 5 \cdot 2^2 \cdot 3,$$

$$12 = \tfrac{1}{7} \cdot \tfrac{1}{11} \cdot 2^2 \cdot 3 \cdot 7 \cdot 11.$$

However, we do not consider the last two of these as acceptable factorizations and the first two do not give us *prime* factors. Factors of integers must be integers, and an integer is not *completely* factored until it is expressed as the product of prime factors.

The expressions we shall learn to factor are polynomials whose terms have integral coefficients. To factor such a polynomial shall mean to express it as the product of two or more other polynomials with integral coefficients.

Examples 14.1.1

1. $x^2 - 4 = (x - 2)(x + 2)$.
2. $x^4 - 16 = (x^2 - 4)(x^2 + 4)$.
3. $x^4 - 16 = (x - 2)(x + 2)(x^2 + 4)$.
4. $x^2 + 6x + 9 = (x + 3)^2$.

5. $x^2 - x - 2 = (x + 1)(x - 2)$.
6. $6x^2 + 13xy - 5y^2 = (3x - y)(2x + 5y)$.

Verify that these are factored correctly.[25]

A polynomial will be called *prime* if it has no factors of the type that we are considering other than plus and minus itself and plus and minus 1. We shall consider a polynomial as completely factored only when all its factors are prime. Example 2 above is not completely factored since the first factor is not prime.

Factoring is essentially recognizing the type of product which is given. When we have $x^2 - 4$ we recognize that this is "the difference of two squares," the difference of x^2 and 2^2, and remember that this is obtained by multiplying the sum and difference, $x + 2$ and $x - 2$, of the two numbers, x and 2. When we have $x^2 + 6x + 9$ we recognize that the first and last terms are the squares of x and 3, respectively, and that the middle term is twice the product of these numbers. Thus $x^2 + 6x + 9 = (x + 3)^2$.

In a sense factoring is guessing. We guess what the factors may be and then check our guess by multiplying the "guessed" factors to see if we obtain the given expression. Factoring is guessing, but it is educated guessing. We do not make wild guesses, but, knowing what possible factors can give a certain product, we restrict our guesses to those possibilities.

14.2 Quadratic Binomials and Trinomials

Our guesses should be made in an orderly manner. In general when we are seeking the factors of a polynomial we should follow the order:

1. Look for a common monomial factor (the HCF) of all the terms of the polynomial; if there is one, take it out:

$$20x^2y - 5y^3 = 5y(4x^2 - y^2).$$

If the coefficient of the highest power is negative, it is generally best to take out the common factor -1:

$$-7x^2 + 2x - 3 = -1(7x^2 - 2x + 3) = -(7x^2 - 2x + 3).$$

2. Examine the expression for the form "the difference of two squares"; if it is of this form, then make use of the fact that $a^2 - b^2 = (a - b)(a + b)$:

$$5y(4x^2 - y^2) = 5y[(2x)^2 - y^2] = 5y(2x - y)(2x + y).$$

3. Examine the expression for the form of a "trinomial perfect square":

$$a^2 + 2ab + b^2 = (a + b)^2$$

or

$$a^2 - 2ab + b^2 = (a - b)^2.$$

In this form the first and last terms must be monomial perfect squares and the middle term must be plus or minus twice the product of the numbers whose squares are the first and last terms:

$$x^2 + 6x + 9 = x^2 + 2\cdot 3\cdot x + 3^2 = (x + 3)^2;$$

$$25x^2 - 70xy + 49y^2 = (5x)^2 - 2(5x)(7y) + (7y)^2 = (5x - 7y)^2.$$

4. If it is none of the first three types, then we examine the expression to see if it is one of the forms

$$x^2 + (b + d)x + bd = (x + b)(x + d)$$

or

$$acx^2 + (ad + bc)x + bd = (ax + b)(cx + d).$$

The first of these is the special case of the second when $a = c = 1$. Of course, a necessary condition that such a quadratic trinomial be of the first form is that the coefficient of x^2 be 1.

In guessing the factors of a quadratic trinomial of the first form we first consider the constant (last) term. The values of b and d must be integers whose product is this term. We raise the question, Can we find factors of this term such that the sum of these factors is the coefficient of the first degree (middle) term? If we can find such factors, then these are the values of b and d.

EXAMPLES 14.2.1

1. Consider $x^2 + 7x + 10$. The factors of 10 are 1 and 10, -1 and -10, 2 and 5, -2 and -5. Of these factors only 2 and 5 have the sum 7. Therefore,

$$x^2 + 7x + 10 = (x + 2)(x + 5).$$

2. Consider $x^2 + 11x + 10$. Again we consider the factors of 10. This time 10 and 1 are the two whose sum is 11. Therefore,

$$x^2 + 11x + 10 = (x + 10)(x + 1).$$

3. Consider $x^2 + x - 2$. The factors of -2 are 1 and -2 and -1 and 2. Of these only -1 and 2 have the sum 1. Therefore,

$$x^2 + x - 2 = (x - 1)(x + 2).$$

4. Consider $x^2 + 3x + 4$. The factors of 4 are 1 and 4, -1 and -4, 2 and 2, -2 and -2. None of these have the sum 3. We say that $x^2 + 3x + 4$ is not factorable (into the kind of factors we are seeking).

Note: If none of the terms of the trinomial are negative, then we need not consider any negative factors of the last term. If the last term is negative, then the factors of the last term must be opposite in sign; if the last term is positive, then the factors of the last term must have the same sign. If the middle term is negative, then at least one factor of the last term must be negative.

Examples 14.2.2

1. For $x^2 + 5x + 6$ we need consider $(x + 1)(x + 6)$ and $(x + 2)(x + 3)$ only.

2. For $x^2 + 5x - 14$ we need consider $(x + 2)(x - 7)$, $(x - 2)(x + 7)$, $(x + 1)(x - 14)$, and $(x - 1)(x + 14)$ only.

3. For $x^2 - 8x - 33$ we need consider $(x + 1)(x - 33)$, $(x - 1)(x + 33)$, $(x + 3)(x - 11)$, and $(x - 3)(x + 11)$ only.

4. For $x^2 - 6x + 8$ we need consider $(x - 1)(x - 8)$ and $(x - 2)(x - 4)$ only.

When we have a quadratic trinomial of the second form we must consider the divisors of both the constant term and the coefficient of the square (first) term. In order to factor successfully it is important to consider these divisors in an orderly manner.

Example 14.2.3

Consider the quadratic trinomial

$$6x^2 + 7x + 2.$$

Comparing it with $acx^2 + (ad + bc)x + bd = (ax + b)(cx + d)$, we note that a and c must be such that their product is 6 and b and d must be such that their product is 2. Furthermore, $ad + bc$ must equal 7.

The possible values of a and c are 6 and 1 or 3 and 2; the possible values of b and d are 1 and 2 or 2 and 1. (As above, we need not consider negative values.) We arrange these possible values of a, b, c, and d as though we were "cross multiplying":

$$\begin{array}{ccc} 6 & & 1 \\ & \times & \\ 1 & & 2, \end{array} \qquad \begin{array}{ccc} 6 & & 2^* \\ & \times & \\ 1 & & 1, \end{array} \qquad \begin{array}{ccc} 3 & & 1^* \\ & \times & \\ 2 & & 2, \end{array} \qquad \begin{array}{ccc} 3 & & 2 \\ & \times & \\ 2 & & 1, \end{array}$$

This corresponds to trying as possible factors:

$$\begin{array}{cccc} 6x+1 & 6x+2^* & 3x+1^* & 3x+2 \\ \text{and} & \text{and} & \text{and} & \text{and} \\ x+2, & x+1, & 2x+2, & 2x+1. \end{array}$$

Since $3 \cdot 1 + 2 \cdot 2 = 7$ we have $6x^2 + 7x + 2 = (3x + 2)(2x + 1)$.

Note: In factoring this quadratic trinomial we need not have considered the two possibilities which are starred. In each of these one of the possible factors is itself divisible by 2. These therefore cannot be the correct factors since the original trinomial $6x^2 + 7x + 2$ is not divisible by 2. The terms of a factor of a polynomial cannot have a common factor unless the terms of the polynomial also have this common factor. Since our first step in factoring is to take out any common factors we should never have to consider as a possible factor an expression whose terms have a common factor.

EXAMPLE 14.2.4

Consider $6x^2 + 15x - 9$. Taking out the common factor 3, we have

$$6x^2 + 15x - 9 = 3(2x^2 + 5x - 3).$$

We now consider $2x^2 + 5x - 3$:

$$\begin{array}{ccc} 2 & & -3 \\ & \times & \\ 1 & & 1, \end{array} \qquad \begin{array}{ccc} 2 & & 1 \\ & \times & \\ 1 & & -3, \end{array} \qquad \begin{array}{ccc} 2 & & -1 \\ & \times & \\ 1 & & 3, \end{array} \qquad \begin{array}{ccc} 2 & & 3 \\ & \times & \\ 1 & & -1. \end{array}$$

Since $(2) \cdot (3) + (1) \cdot (-1) = 5$ (the third possibility), we have

$$2x^2 + 5x - 3 = (2x - 1)(x + 3)$$

and

$$6x^2 + 15x - 9 = 3(2x - 1)(x + 3).$$

EXERCISE 14.2.5

Factor each of the following; for those problems for which values of the letters are given, check the product of the values of the factors with the value of the polynomial:

1. $6x^4 + 9x^2$.
2. $5a^2 - 30ab$.
3. $9x - 3x^2$; $x = 2$.
4. $4p^2 - 16p$; $p = 2$.
5. $15a^2x^4 + 30a^3x^2$.
6. $p^2q^3 - p^3q^2$.
7. $3(x + y) + 2x(x + y)$.
8. $2x(a + b) + 6y(a + b)$.
9. $a(a + b) + b(a + b)$.
10. $2^5x^3y^6 + 2^6x^4y^3$.
11. $x^2(x + y)^2 + y^2(x + y)^2$.
12. $x^2y(x + y)^2 + xy^2(x + y)^2$.

13. $9pq^2r^2 + 27p^3q^2 - 81pq^3r^3$.
14. $25ab^8c^3 - 75a^2b^5 + 125b^3c^3$.
15. $2^7r^2s^2t^2 + 2^5r^5s^2t^3 + 4r^3s^2t^2$.

16. $x^2 - 9$; $x = 5$.
17. $x^4 - 9x^2$; $x = -3$.
18. $a^2b^2 - a^4$; $a = 8$, $b = 9$.
19. $16x^2 - 1$; $x = 2$.

20. $9p^2 - 49q^2$; $p = 159$, $q = 68$.
21. $27m^2n^2 - 12n^4$; $m = -2$, $n = 3$.

22. $45a^3b^2 - 20ac^2$.
23. $(x^2)^2 - (y^2)^2$.
24. $r^4 - s^4$; $r = -s = 5$.
25. $a^4 - 16b^4$; $a = b = \frac{1}{2}$.
26. $80x^4 - 5y^4$.
27. $x^2 - 8x + 16$.
28. $p^2 + 10p + 25$; $p = 2$.
29. $4a^2 + 12a + 9$; $a = -1$.

30. $9a^2 + 12a + 4$; $a = -1$.
31. $x^2 - 10xy - y^2$; $x = y = 2$.

32. $25x^2 - 10xy - y^2$.
33. $25c^2 + 20cd + 4d^2$.
34. $36m^2 - 60mn + 25n^2$.
35. $(x^2)^2 - 2x^2y^2 + (y^2)^2$.
36. $a^4 - 2a^2b^2 + b^4$.
37. $x^4 - 8x^3 + 16x^2$.
38. $4p^4 + 40p^3 + 100p^2$.
39. $9a^2b^2 - 12ab^2 + 4b^2$.
40. $400x^2 - 600x + 225$.
41. $400x^2 + 480x + 144$.
42. $100x^2y^2 + 120xy^2 + 36y^2$.
43. $100x^2y^2 + 120xy^3 + 36y^4$.
44. $18a^2b^3c - 48ab^4c + 32b^5c$.
45. $x^2 + 7x + 12$.
46. $x^2 + 8x + 12$; $x = 3$.
47. $x^2 + 13x + 12$; $x = 3$.
48. $x^2 - 7x + 12$; $x = 3$.
49. $x^2 - 8x + 12$; $x = 3$.
50. $x^2 - 13x + 12$; $x = 3$.
51. $x^2 + 7x$; $x = 3$.
52. $x^2 + 7x + 6$.
53. $x^2 + 7x + 10$.
54. $x^2 + 7x + 8$.
55. $x^2 + 7x - 18$.

56. $x^2 + 7x - 30.$
57. $x^2 - 7x + 8.$
58. $x^2 - 7x - 18.$
59. $x^2 - 7x - 30.$
60. $a^2 - 11a - 12.$
61. $a^2 - 3a + 18.$
62. $p^2 - 13p + 36.$
63. $r^2 - r - 20.$
64. $m^2 + 10m + 21.$
65. $c^2 + 10c - 75.$
66. $x^2 + ax - 42a^2.$
67. $y^2 - 18xy + 45x^2.$
68. $p^2 - 26pq + 169q^2.$
69. $9a^2 - 63a + 108.$
70. $4a^2b^2 - 28ab^2 - 32b^2.$
71. $x^4 - 11x^3 + 12x^2.$
72. $x^4 - 3x^2 - 4; x = 2.$
73. $5x^4 - 120x^2 - 125; x = -1.$

74. $x^4 - 5x^2 + 4; x = -3.$
75. $x^4 - 13x^2 + 36; x = -3.$
76. $2x^2 + 3x + 1; x = -\frac{1}{2}.$
77. $3x^2 - 8x + 4; x = \frac{2}{3}.$

78. $10x^2 + 19x + 6.$
79. $6x^2 + 5x - 6.$
80. $6x^2 - 5x + 6.$
81. $12x^2 - 29x + 15.$
82. $12x^2 - 11x - 15.$
83. $12x^2 + 8x + 15.$
84. $2x^2 + 12x + 10.$
85. $15x^2 - 30x + 15.$
86. $15x^2 + 50x + 15.$
87. $-x^2 + 6x + 8.$
88. $-2x^2 + 3x + 5.$
89. $-16x^2 - 12x + 18.$
90. $50x - 7x^2 - 7.$
91. $58x - 21 - 21x^2.$
92. $4p^2 - 9q^2.$
93. $(x + y)^2 - a^2.$
94. $p^2 - (q + r)^2.$
95. $x^2 + 2xy + y^2 - a^2.$
96. $p^2 - q^2 - 2qr - r^2.$
97. $(a^2 - b^2)^2 - c^2.$
98. $(a - b)^4 - c^4.$
99. $a^2 - x^2 + 2xy - y^2.$
100. $(x + 1)^2 + 2(x + 1) + 1.$
101. $(3x^2 - 2)^2 - x^2.$
102. $(5x^2 - 3)^2 - 4x^2.$
103. $36(x^2 - 1)^2 - 25x^2.$
104. $25(x^2 - 2)^2 - 529x^2.$

Solve the following equations:

105. $x^2 - 4 = 0.$
106. $x^3 - 9x = 0.$
107. $6x^2 - 24 = 0.$
108. $x^2 - 6x + 9 = 0.$
109. $x^2 + 6x + 9 = 0.$
110. $9x^2 - 12x + 4 = 0.$
111. $15x^2 - 62x + 63 = 0.$
112. $x^4 - 5x^2 + 4 = 0.$
113. $x^4 - 13x^2 + 36 = 0.$
114. $(3x^2 - 2)^2 - x^2 = 0.$
115. $(5x^2 - 3)^2 - 4x^2 = 0.$
116. $36(x^2 - 1)^2 - 25x^2 = 0.$
117. $25(x^2 - 2)^2 - 529x^2 = 0.$
118. $9x^4 - 13x^2 + 4 = 0.$
119. $25x^4 - 34x^2 + 9 = 0.$
120. $36x^4 - 97x^2 + 36 = 0.$
121. $25x^4 - 629x^2 + 100 = 0.$
122. $9x^4 - 13x^3 + 4x^2 = 0.$
123. $36x^4 - 97x^3 + 36x^2 = 0.$
124. $25x^4 - 629x^3 + 100x^2 = 0.$

14.3 HCF and LCM of Polynomials

From our previous experience in finding the HCF and the LCM of two or more quantities we realize that the methodical way to find these numbers is to make use of the prime factors of the given quantities. (See 11.4 and 11.5.) Thus the HCF of $36a^2b^3$ and $120a^4b^2$ is obtained from

$$36a^2b^3 = 2^2\cdot 3^2\cdot a^2\cdot b^3$$

and

$$120a^4b^2 = 2^3\cdot 3\cdot 5\cdot a^4\cdot b^2$$

by picking out the *lowest* power of each prime factor which occurs in the two quantities. In this case the HCF is $2^2\cdot 3\cdot a^2\cdot b^2 = 12a^2b^2$.

The LCM is obtained by picking out the *highest* power of each prime factor. In this case the LCM is $2^3\cdot 3^2\cdot 5a^4b^3 = 360a^4b^3$.

We have used the HCF in picking out the common factors of the terms of a polynomial and in reducing fractions to lowest terms. We have used the LCM in adding and subtracting fractions. We found the LCM of the denominators and called it the LCD. We can now apply these concepts to polynomials.

Examples 14.3.1

1. To find the HCF and LCM of $x^2 - 4$ and $x^2 - x - 2$ we write

$$x^2 - 4 = (x - 2)(x + 2),$$

$$x^2 - x - 2 = (x - 2)(x + 1).$$

Then the HCF is $x - 2$ and the LCM is $(x - 2)(x + 2)(x + 1)$.

2. For $x^4 - 16$ and $x^2 - 4x + 4$ we have

$$x^4 - 16 = (x - 2)(x + 2)(x^2 + 4),$$

$$x^2 - 4x + 4 = (x - 2)^2.$$

Then the HCF is $x - 2$ and the LCM is $(x - 2)^2(x + 2)(x^2 + 4)$.

3. For $(x^2 + x - 2)^2$ and $x^3 - 2x^2 + x$ and $(x^4 + 2x^3 - 3x^2)^2$ we have

$$(x^2 + x - 2)^2 = (x + 2)^2(x - 1)^2,$$

$$x^3 - 2x^2 + x = x(x - 1)^2,$$

$$(x^4 + 2x^3 - 3x^2)^2 = x^4(x + 3)^2(x - 1)^2.$$

Then the HCF is $(x-1)^2$ and the LCM is $x^4(x+2)^2(x-1)^2(x+3)^2$.

4. To reduce the fraction $\dfrac{x^2+5x+6}{x^2-9}$ to lowest terms, we write

$$\frac{x^2+5x+6}{x^2-9}=\frac{(x+3)(x+2)}{(x+3)(x-3)}=\frac{x+2}{x-3}.$$

5. To write the fractions $\dfrac{x}{4x^2-25}$ and $\dfrac{x+2}{2x^2+7x+5}$ with the LCD, we write

$$\frac{x}{4x^2-25}=\frac{x}{(2x-5)(2x+5)}$$

and

$$\frac{x+2}{2x^2+7x+5}=\frac{x+2}{(2x+5)(x+1)}.$$

The LCD is $(2x-5)(2x+5)(x+1)$ so that

$$\frac{x}{(2x-5)(2x+5)}=\frac{x(x+1)}{(2x-5)(2x+5)(x+1)}$$

and

$$\frac{x+2}{(2x+5)(x+1)}=\frac{(x+2)(2x-5)}{(2x-5)(2x+5)(x+1)}.$$

The sum of these fractions is then:

$$\begin{aligned}\frac{x(x+1)}{(2x-5)(2x+5)(x+1)}+\frac{(x+2)(2x-5)}{(2x-5)(2x+5)(x+1)}&=\frac{x(x+1)+(x+2)(2x-5)}{(2x-5)(2x+5)(x+1)}\\&=\frac{x^2+x+2x^2-x-10}{(2x-5)(2x+5)(x+1)}\\&=\frac{3x^2-10}{(2x-5)(2x+5)(x+1)}.\end{aligned}$$

6. $$\frac{2x+4}{2x^2-x-10}-\frac{3x}{3x^2+5x}$$

$$= \frac{2x+4}{(2x-5)(x+2)}-\frac{3x}{x(3x+5)}$$

$$= \frac{(2x+4)(3x+5)x-3x(2x-5)(x+2)}{x(2x-5)(x+2)(3x+5)}$$

$$= \frac{(6x^3+22x^2+20x)-(6x^3-3x^2-30x)}{x(2x-5)(x+2)(3x+5)}$$

$$= \frac{6x^3+22x^2+20x-6x^3+3x^2+30x}{x(2x-5)(x+2)(3x+5)}$$

$$= \frac{25x^2+50x}{x(2x-5)(x+2)(3x+5)}$$

$$= \frac{25x(x+2)}{x(2x-5)(x+2)(3x+5)}$$

$$= \frac{25}{(2x-5)(3x+5)}$$

When operating with fractions always remember to start by reducing each fraction to lowest terms if it is not already in lowest terms.

EXERCISE 14.3.2

Find the HCF and LCM of:

1. $3x^3+3x^2$ and $6x^4-6x^3$.
2. $14x^3-28x^2$ and $21x^4-42x^3$.
3. $p^2q^3-p^3q^2$ and $p^2q^3+p^3q^2$.
4. $p^2q^3-p^3q^2$ and $p^4q^2-p^2q^4$.
5. $p^2q^3+p^3q^2$ and $p^4q^2-p^2q^4$.
6. p^2-q^2 and $p^2-2pq+q^2$.
7. $9a^2-49b^2$ and $9a^2+42ab+49b^2$.
8. $9a^2-49b^2$ and $9a^2-42ab+49b^2$.
9. x^4-16 and x^4+4x^2.
10. x^4-16 and x^3-4x.

11. $x^2 - x - 2$ and $x^2 - 4$.
12. $4x^3 + 4x^2 - 8x$ and $8x^3 - 32x$.
13. $m^2 + 10m + 21$ and $m^2 + 6m + 9$.
14. $m^2 - 13m + 12$ and $m^2 - 10m - 24$.
15. $9a^2b^2 - 12ab^2 + 4b^2$ and $9a^2b^4 - 4b^4$.
16. $12x^2 - 29x + 15$ and $12x^2 - 11x - 15$.
17. $12x^2 + 8x - 15$ and $36x^2 - 25$.
18. $(3x^2 - 2)^2 - x^2$ and $(3x^2 - 4)^2 - 16x^2$.
19. $4(x^2 - 1)^2 - 9x^2$ and $4(x^2 - 2)^2 - 49x^2$.
20. $(x^2 - x - 6)^2$ and $x^4 - 13x^2 + 36$.

Reduce to lowest terms:

21. $\dfrac{7x + 7y}{x^2 + xy}$.

22. $\dfrac{ax}{a^2x^2 - ax}$.

23. $\dfrac{3x^2 - 6xy}{2x^2y - 4xy^2}$.

24. $\dfrac{x^2 + x - 2}{x^2 - 4}$.

25. $\dfrac{4x^3 + 4x^2 - 8x}{8x^3 - 32x}$.

26. $\dfrac{m^2 + 10m + 21}{m^2 + 6m + 9}$.

27. $\dfrac{a^2 - 13a + 12}{a^2 - 10a - 24}$.

28. $\dfrac{3x^2 + 8x + 5}{3x^2 - x - 10}$.

29. $\dfrac{9x^2 - 25}{3x^2 - 11x + 10}$.

30. $\dfrac{a^2b^2 - ab}{a^2b^2 - 1}$.

31. $\dfrac{x^4 - 16}{x^4 + 4x^2}$.

32. $\dfrac{12x^2 - 29x + 15}{12x^2 - 11x - 15}$.

33. $\dfrac{24x^3 + 16x^2 - 30x}{36x^4 - 25x^2}$.

34. $\dfrac{(x^2 - 4)(x^2 - 9)}{(x^2 + x - 6)(x^2 - x - 6)}$.

35. $\dfrac{(4x^2 - 9)(9x^2 - 4)}{(4x^2 - 12x + 9)(9x^2 + 12x + 4)}$.

36. $\dfrac{(16x^2 - 12x + 2)(16x^2 - 28x + 12)}{(16x^2 - 16x + 3)(64x^2 - 4)}$.

37. $\dfrac{(9x^2 + 15x + 6)(9x^2 + 27x + 20)}{(9x^2 + 24x + 16)(9x^2 + 21x + 10)}$.

38. $\dfrac{(3x^4 + 9x^3y + 6x^2y^2)(x^2y^2 - y^4)}{(x^2y^2 + y^4)(x^4 + x^3y - 2x^2y^2)}$.

39. $$\frac{(x^2-4)(x^2-9)(x^2-16)}{(x^2-6x+8)(x^2+6x+8)(x^2-6x+9)}.$$

40. $$\frac{(2x^2-x-6)(3x^2+11x+6)(25x^2-10x+1)}{(10x^2+13x-3)(15x^2+13x+2)(x^2-5x-6)}.$$

Perform the indicated operations and reduce to lowest terms:

Group A

41. $$\frac{4xy}{x+3}\cdot\frac{3x^2+9x}{16y^2}.$$

42. $$\frac{a+b}{2a+b}\cdot\frac{4a^2-b^2}{a^2-b^2}.$$

43. $$\frac{a^2+5a+6}{a^2-1}\cdot\frac{a^2-2a-3}{a^2-9}.$$

44. $$\frac{2p+1}{2p+2}\cdot\frac{2p^2-p-3}{4p^2-4p-3}.$$

45. $$\frac{p^2-q^2}{p^2q^2}\cdot\frac{p^4}{p+q}\cdot\frac{q^2}{p^2-pq}.$$

46. $$\frac{s^2+s}{2s+1}\cdot\frac{10s+5}{st-t}\cdot\frac{s^2t-st}{10s+10}.$$

47. $$\frac{2x^2+9x-5}{2x^2+3x+1}\cdot\frac{3x^2+x-2}{4x^2-8x+3}\cdot\frac{4x^2-4x-3}{3x^2+7x-6}.$$

48. $$\frac{24x^3+16x^2-30x}{12x^2-11x-15}\cdot\frac{18x^2-21x-15}{8x^4+16x^3+6x^2}\cdot\frac{64x^3-36x}{108x^2-36x-45}.$$

49. $$\frac{x^2-3x-10}{x^2+2x-35}\div\frac{x^2+9x+14}{x^2+4x-21}.$$

50. $$\frac{x^2+4x+4}{x^2-9}\div\frac{x^2-4}{x^2-6x+9}.$$

51. $$\frac{4x^2-9}{4x-4}\div\left(\frac{2x^2-x-3}{4x-7}\cdot\frac{2x+3}{4x^2-4}\right).$$

52. $$\left(\frac{4x^2-9}{4x-4}\div\frac{2x^2-x-3}{4x-7}\right)\cdot\frac{2x+3}{4x^2-4}.$$

Group B

53. $\dfrac{2}{x+2} + \dfrac{4}{3x+6} - \dfrac{5}{6x+12}.$

54. $\dfrac{1}{5x-5} + \dfrac{4}{2x-2} - \dfrac{33}{15x-15}.$

55. $\dfrac{1}{x+1} + \dfrac{2}{x+2} + \dfrac{3}{x+3}.$

56. $\dfrac{4}{y-2} + \dfrac{7}{y-1} - \dfrac{5}{3y+1}.$

57. $\dfrac{1}{(x+y)^2} + \dfrac{1}{x^2-y^2}.$

58. $\dfrac{4}{x^2-4} - \dfrac{3}{x^2-x-2}.$

59. $\dfrac{1}{p+q} - \dfrac{1}{p-q} + \dfrac{2p}{p^2-q^2}.$

60. $\dfrac{1}{x+1} + \dfrac{3}{x^2+2x+1} - \dfrac{2}{x^2-1}.$

61. $\dfrac{x^2}{x^4-3x^3+2x^2} + \dfrac{1}{x^2-5x+6}.$

62. $\dfrac{16}{4x^2-16y^2} + \dfrac{1}{x^2-5xy+6y^2}.$

63. $\dfrac{x^2+1}{x^4-1} - \dfrac{1}{x^2+x-2}.$

64. $\dfrac{x+3}{(x^2+4x+3)(x-2)} + \dfrac{x-1}{(x^2-3x+2)(x-3)}$

$- \dfrac{x-2}{(x^2-5x+6)(x+1)}.$

65. $\dfrac{x+2y}{(x^2+xy-2y^2)(x+y)} - \dfrac{x+y}{(x^2+3xy+2y^2)(x-y)}$

$+ \dfrac{x-y}{(x^2-y^2)(x+2y)}.$

66. $\dfrac{1}{x^2-3x+2} + \dfrac{3}{x^2-x-2} - \dfrac{2}{x^2-1}.$

67. $\dfrac{4}{x^2+6x+8} - \dfrac{2}{x^2+7x+12} + \dfrac{x}{x^2+5x+6}.$

68. $$\frac{5x+5}{(x^2+3x+2)(x-3)}+\frac{4x-8}{(x+3)(x^2-4x+4)}-\frac{2x-6}{(x^2+x-12)(x+3)}.$$

Group C

69. $$\frac{1}{5x-5}-\frac{1}{3x-3}+\frac{1}{1-x}.$$

70. $$\frac{2}{10-5x}+\frac{5}{2-x}-\frac{7}{2x-4}.$$

71. $$\frac{1}{2+x}+\frac{1}{x-2}-\frac{1}{x^2-4}.$$

72. $$\frac{1}{x^2+x-6}+\frac{1}{12-x-x^2}.$$

73. $$\frac{x^2+x+1}{x+1}-\frac{x^2-x+1}{x-1}.$$

74. $$\frac{x^2+2x+4}{x+2}+\frac{x^2-2x+4}{2-x}.$$

75. $$\frac{x+1}{x-1}\cdot\left(\frac{1}{x+1}+\frac{1}{x-1}\right).$$

76. $$\frac{x^2+3x+2}{x^2-3x+2}\cdot\left(\frac{x-1}{x^2+4x+3}-\frac{x-2}{x^2-x-6}\right).$$

77. $$\frac{\dfrac{x-1}{x+1}-\dfrac{x+1}{x-1}}{\dfrac{2}{x^2-1}}.$$

78. $$\frac{\dfrac{x}{x^2+2x-35}+\dfrac{1}{x^2+9x+14}}{\dfrac{x-3}{x^2-3x-10}}.$$

79. $$\frac{\dfrac{a^2 + ab + b^2}{a + b} - \dfrac{a^2 - ab + b^2}{a - b}}{\dfrac{1}{a + b} - \dfrac{1}{a - b}}.$$

80. $$\frac{\dfrac{4x^2 - 6x + 9}{2x - 3} - \dfrac{4x^2 + 6x + 9}{2x + 3}}{\dfrac{x}{2x - 3} - \dfrac{1}{2x + 3}}.$$

14.4 The Sum and Difference of Two Cubes

In doing Problem 79 of Exercise 14.3.2 you had occasion to perform the multiplications

$$(a + b)(a^2 - ab + b^2) = a^3 + b^3$$

and

$$(a - b)(a^2 + ab + b^2) = a^3 - b^3.$$

These are two more "type products" with which it is handy to be familiar. (See 13.2.1.) Reading them from right to left gives us methods of factoring "the sum and difference of two cubes."

14.4.1 $$a^3 + b^3 = (a + b)(a^2 - ab + b^2).*$$

14.4.2 $$a^3 - b^3 = (a - b)(a^2 + ab + b^2).*$$

It will help you to remember these forms if you note that the *sum* of two cubes has as a factor the *sum* of the numbers which are cubed and that the *difference* of two cubes has as a factor the *difference* of the numbers which are cubed. Further, each pair of factors has *just one minus sign.* If the minus sign is not in the first factor, then it must be in the second and vice versa.

In using these forms we first write down the binomial factor and then construct the trinomial factor from it. The first term of the trinomial is the square of the first term of the binomial, the second term of the trinomial is the product, with opposite sign, of the two terms of the binomial, and the third term is the square of the second term of the binomial.

---

* It can be shown that expressions of the forms $a^2 - ab + b^2$ or $a^2 + ab + b^2$ cannot, in general, be factored into factors of the type that we are considering. In certain cases, as when $a = x^{2m}$ and $b = y^{2m}$ (m a positive integer), $a^2 + ab + b^2 = x^{4m} + x^{2m}y^{2m} + y^{4m}$ can be factored further. This will be discussed in 14.6.

Examples 14.4.3

1. $a^3 + 8 = a^3 + 2^3$
$$= (a + 2)(a^2 - 2a + 2^2)$$
$$= (a + 2)(a^2 - 2a + 4).$$

2. $8x^3 - 27 = (2x)^3 - 3^3$
$$= (2x - 3)[(2x)^2 + 3\cdot 2x + 3^2]$$
$$= (2x - 3)(4x^2 + 6x + 9).$$

3. $125x^3 + 8y^3 = (5x)^3 + (2y)^3$
$$= (5x + 2y)(25x^2 - 10xy + 4y^2).$$

Note that the second term of the trinomial factor is ab, *not* $2ab$, as it is in a trinomial perfect square.

Exercise 14.4.4

Factor:

1. $x^3 - 1.$
2. $x^3 + 1.$
3. $8a^3 + 1.$
4. $8a^3 - 1.$
5. $a^3 + 8.$
6. $a^3 - 8.$
7. $27x^3 + 8.$
8. $27x^3 - 8.$
9. $8p^3 + 125q^3.$
10. $8p^3 - 125q^3.$
11. $x^3y^3 - 1.$
12. $1 + x^3y^3.$
13. $27a^3b^3 - 125c^3.$
14. $125a^3b^3 + 27c^3.$
15. $x^3 + (x + 1)^3.$
16. $8x^3 + (x - 1)^3.$
17. $(x + 1)^3 - (x - 1)^3.$
18. $(x + 1)^3 + (x - 1)^3.$
19. $(2x + 1)^3 - x^3.$
20. $(x + 2)^3 + (x - 2)^3.$
21. $(2x + 3)^3 + (3x + 2)^3.$
22. $(2x + 3)^3 - (3x + 2)^3.$
23. $8(x + 2)^3 - 27(x + 3)^3.$
24. $125(2x + 1)^3 + 27(3x - 1)^3.$
25. $x^6 - y^6.$ [Hint: $x^6 = (x^3)^2$.]
26. $x^6 - 64.$
27. $x^6 - 64x^3.$
28. $x^6 - 7x^3 - 8.$
29. $8x^6 + 63x^3 - 8.$
30. $8x^6 - 65x^3 + 8.$
31. $8x^6 - x^3.$
32. $8x^3 - 64x^6.$
33. $9x^8 - 72x^5.$
34. $6x^6 + 156x^3 - 162.$

14.5 Factoring by Grouping Terms

Sometimes the terms of a polynomial may be grouped in such a way that we can see that the resulting expression is one of the forms we have considered. Problem 8 of Exercise 14.2.5 is of this type. If this problem originally had been written as

$$2ax + 2bx + 6ay + 6by,$$

and we had grouped the first two and the last two terms, we would have obtained

$$(2ax + 2bx) + (6ay + 6by) = 2x(a + b) + 6y(a + b),$$

and we see that we then can take out the common factor $a + b$, obtaining

$$(a + b)(2x + 6y).$$

Again, in Problem 95 of Exercise 14.2.5 we have $x^2 + 2xy + y^2 - a^2$, which can be written

$$(x^2 + 2xy + y^2) - a^2 = (x + y)^2 - a^2,$$

and we see that this is of the form "the difference of two squares."

EXERCISE 14.5.1

Factor:

1. $ax - bx + ay - by.$
2. $a^2 + ab + ac + bc.$
3. $ax - bx - ay + by.$
4. $ax + by - bx - ay.$
5. $x^3 + 2x^2 + x + 2.$
6. $x^3 + x - 2x^2 - 2.$
7. $x^3 - x + 2x^2 - 2.$
8. $8x^6 + 16x^5 - 32x^4 - 64x^3.$
9. $4x^3 - 12x^2 - 9x + 27.$
10. $9x^3 + 18x^2y - 4xy^2 - 8y^3.$
11. $x^2 - 4x + 4 - y^2.$
12. $9x^3 - 18x^2y - 4xy^2 + 8y^3.$
13. $9x^3 + 6x^2 + x - 4xy^2.$
14. $4x^2 - y^2 - 2y - 1.$
15. $(x^4 + 2x^2y^2 + y^4) - x^2y^2.$
16. $(x^4 + 4x^2 + 4) - 4x^2.$

17. $x^4 + 16x^2y^2 + 64y^4 - 16x^2y^2.$

18. $16x^4 + 72x^2 + 81 - 36x^2.$
19. $x^4 + ax^3 + x + a.$
20. $x^4 - ax^3 + x - a.$
21. $x^4 + ax^3 + a^3x + a^4.$
22. $x^4 - ax^3 + a^3x - a^4.$
23. $x^4 + ax^3 - a^3x - a^4.$
24. $x^4 - ax^3 - a^3x + a^4.$

14.6 Factoring of Certain Fourth-Degree Polynomials

Problem 25 of Exercise 14.4.4 asked us to factor $x^6 - y^6$. The hint given was that $x^6 = (x^3)^2$. Using this hint, we see that this problem can be considered "the difference of two squares." We obtain

$$x^6 - y^6 = (x^3)^2 - (y^3)^2 = (x^3 - y^3)(x^3 + y^3).$$

These last factors can be factored further so that we finally have

$$x^6 - y^6 = (x - y)(x^2 + xy + y^2)(x + y)(x^2 - xy + y^2).$$

But, of course, we may also write $x^6 = (x^2)^3$, and we then have

$$\begin{aligned} x^6 - y^6 = (x^2)^3 - (y^2)^3 &= (x^2 - y^2)(x^4 + x^2y^2 + y^4) \\ &= (x - y)(x + y)(x^4 + x^2y^2 + y^4). \end{aligned}$$

The factors $x - y$ and $x + y$ are the same as two of the factors we obtained by the first method. But the trinomial factor we obtain by this method does not look like any polynomial we have previously factored.

When a polynomial is factored into the kind of factors that we are considering, it must always yield the same factors no matter what method is used. (This has not been proved here but we ask you to accept this statement.). The only possible difference is the order of writing down the factors. You surely agree that

$$30 = 2\cdot 3\cdot 5$$

and

$$30 = 5\cdot 2\cdot 3$$

should not be considered as two different factorings of 30. There must be a way to see that

$$x^4 + x^2y^2 + y^4 = (x^2 - xy + y^2)(x^2 + xy + y^2).$$

To see this we write

$$x^4 + x^2y^2 + y^4 = x^4 + 2x^2y^2 + y^4 - x^2y^2.$$

(We have added x^2y^2 to and also subtracted x^2y^2 from this polynomial. Certainly adding and subtracting the same number to a quantity does not change the value of the quantity.) Now

$$\begin{aligned} x^4 + 2x^2y^2 + y^4 - x^2y^2 &= (x^2 + y^2)^2 - (xy)^2 \\ &= [(x^2 + y^2) - xy][(x^2 + y^2) + xy] \\ &= (x^2 + y^2 - xy)(x^2 + y^2 + xy), \end{aligned}$$

and we have the result we should expect.

Exercise 14.6.1

Factor:

1. $x^4 + 4x^2 + 16$.
2. $x^4 + 25x^2y^2 + 625y^4$.
3. $16x^4 + 4x^2 + 1$.
4. $81x^4 + 9x^2 + 1$.
5. $16x^4 + 8x^3 + x^2$.
6. $16x^4 + 64x^3 + 64x^2$.
7. $x^4 + 4$.
8. $x^4 + 324y^4$.
9. $x^4 - 7x^2 + 9$.
10. $x^4 + x^2 + 25$.
11. $x^4 - x^2 + 16$.
12. $4x^4 - 29x^2 + 25$.
13. $9x^4 + 8x^2y^2 + 16y^4$.
14. $9x^4 - 49x^2y^2 + 16y^4$.
15. $x^6 - 64y^6$. (Two methods.)
16. $729x^6 - y^6$.
17. $729x^6 - 64$.
18. $x^6 - 15{,}625$.

15

Functions and Graphs

15.1 Functions

In many fields of study there are related phenomena. In physics there is a relationship between the distance a stone falls and the time during which it falls.* In economics there is a relationship between the price of pork and the number of hogs raised. In English there is a relationship between the form of the verb in a sentence and the subject of the sentence. In genetics there is a relationship between the color of a child's hair and the colors of its parents' hair. In photography there is a relationship between the density of a print and the time of development. In medicine there is a relationship between the occurrence of disease and the presence of bacteria in the blood.

In each of these cases we can determine something about the first of the two items from knowledge of the second. We say that the first item depends upon or is a *function* of the second. When the related phenomena can be expressed in terms of quantities, mathematics frequently enters the picture.

Suppose a group of people propose to play a sequence of singles tennis matches in such a way that each person plays each of the others. How many matches must be played?

If there are just two people there will, of course, be just one match. If there are three people there will be three matches. For, if the people are Tom, Dick, and Harry, we shall have Tom vs. Dick, Tom vs. Harry, Dick vs. Harry. If there are four people, there will be six matches. (Can you show this?)[26] If there are five people, there will be ten matches. We can represent these facts, and some additional ones, by a table:

---

* At one time it was believed that there is a relationship between the time it takes a stone to fall a given distance and the weight of the stone. Galileo (1564–1642) proved, by his famous experiment from the Leaning Tower of Pisa, that this is not correct.

15.1.1										
	Number of players	2	3	4	5	6	7	8	9	10, etc.
	Number of matches	1	3	6	10	15	21	28	36	45, etc.

Here we have a definite pairing between two sets of numbers, the number of players and the number of matches. The number of matches depends upon or is a function of the number of players. Given the number of players, we can determine from the table, or the table extended, the number of matches. It is convenient (but irrelevant to the fact that the number of matches is a function of the number of players) that we can find a simple formula which will give the number of matches if we know the number of players. If m is the number of matches and n the number of players, then $m = \frac{n(n-1)}{2}\cdot$ You can verify this from the table. Can you show that it will be correct for any value of n?[27] What kind of number must n be?[28]

If we have written a long letter and suspect that one 3-cent stamp will not represent enough postage, we take the letter to the post office and have the clerk weigh it. He may tell us that it requires 9 cents postage. The amount of postage is "3 cents for each ounce or fraction thereof." Each weight (in ounces) is paired with a number of cents. If we let p represent the amount of postage and w the weight in ounces, then

15.1.2
$$\begin{aligned} p &= 3 \quad &\text{if} \quad 0 < w \leqq 1 \\ p &= 6 \quad &\text{if} \quad 1 < w \leqq 2 \\ p &= 9 \quad &\text{if} \quad 2 < w \leqq 3 \\ p &= 12 \quad &\text{if} \quad 3 < w \leqq 4 \end{aligned}$$

etc.

Again it is irrelevant, but convenient, that mathematicians have invented a symbol for this kind of function. If we let $[x]$

denote the integer which is not greater than x but is greater than $x - 1$,* then

15.1.3 $$p = 3[w + 1].\dagger$$

The same relationship between p and w is given by the phrase in quotes above, the sequence of statements (15.1.2), or the equality (15.1.3). The important thing is that for each value of w there is a corresponding value of p. Of course, there is no sense in considering negative or zero values of w and postal regulations state that the value of w may not be greater than 1120. Thus a complete statement would be: Each value of w greater than zero but not greater than 1120 has been paired with a value of p.

> 15.1.4 DEFINITION: A variable is said to be a *function* of another variable, if a relation or correspondence or pairing or rule exists such that whenever a value of the second variable is given a value of the first is determined. Since the first variable *depends* upon the second variable, it, the first variable, is called the *dependent variable* and the second is called the *independent variable.*

We have seen that the values which a variable may assume sometimes are determined by the kind of quantity it represents. In our example involving players and tennis matches the variable n, representing the number of players, may take on only integer values greater than or equal to 2. The variable m, representing the number of matches, is still more restricted. We say that the variable n *ranges over* the positive integers greater than or equal to 2 or the *range* of n is the positive integers not less than 2. The range of w, in our postal example, is any number greater than zero and not greater than 1120; or

$$0 < w \leqq 1120.$$

The range of p is over those positive integers which are divisible by 3 and not greater than 3360.

---

* $[3/2] = 1$, since 1 is the integer which is not greater than $3/2$ but is greater than $3/2 - 1 = 1/2$.

$[7/3] = 2$, since 2 is the integer which is not greater than $7/3$ but is greater than $7/3 - 1 = 4/3$.

$[4] = 4$, since 4 is the integer which is not greater than 4 but is greater than $4 - 1 = 3$.

† If $w = 3/2$, then $p = 3[3/2 + 1] = 3 \cdot 2 = 6$.

Exercise 15.1.5

In each of the following state the functional relationship in words and, if possible, by means of a formula. State which is the dependent and which is the independent variable. If possible state the range of each variable.

1. Jimmy's age on his next birthday is a function of his present age.
2. If steak costs 92 cents a pound, then the cost of a steak is a function of its weight.
3. The amount deductible for dependents on an income tax form is a function of the number of dependents.
4. The amount a family with income of $4500 has left for recreation is a function of the amount they must spend for necessities.
5. The area of a square is a function of the length of its sides.
6. The volume of a cube is a function of the length of its edges.
7. The perimeter of a square is a function of the length of its sides.
8. The surface area of a cube is a function of the length of its edges.
9. The area of a circle is a function of its radius.
10. The volume of a vertical cylindrical tank 24 inches in radius is a function of its height.
11. The temperature in degrees Fahrenheit is a function of the temperature in degrees centigrade. (You may have to look this up!)
12. Your income tax is a function of your net income.
13. The number of days in a calendar year is a function of the year.
14. The number of miles per gallon of gasoline obtained from a certain car is a function of the speed at which it is driven.
15. If an open box is to be made from a piece of tin 12 inches long and 8 inches wide by cutting equal squares from the corners and then bending up the edges, the volume of the resulting box is a function of the length of the sides of the squares.
16. The height of a building is a function of the length of its shadow at noon.
17. The number of hairs on a man's head is a function of his social security number.

15.2 Functional Notation

Suppose we have the expression

$$3x^2 - 2.$$

This means "whatever number is substituted for x, square that number, multiply the result by 3, and then subtract 2." The resulting number corresponds to or is paired with the number which we substituted for x. Thus, if we take x to be 2, we find the corresponding or paired number to be $3 \cdot 2^2 - 2 = 10$; if we take x to be -1, we find the corresponding or paired number to be $3(-1)^2 - 2 = 1$. We see that the expression $3x^2 - 2$ is a function of x; for a number x there is a corresponding or paired number $3x^2 - 2$. At times it is of advantage to indicate this by writing

$$f(x) = 3x^2 - 2.$$

Then x is the independent variable and $f(x)$ is the dependent variable.

For example,

$$f(3) = 3 \cdot 3^2 - 2 = 25; \text{ thus 25 is paired with 3.}$$

$$f(1) = 3 \cdot 1^2 - 2 = 1; \text{ thus 1 is paired with 1.}$$

$$f(0) = 3 \cdot 0^2 - 2 = -2; \text{ thus } -2 \text{ is paired with 0.}$$

$$f\left(\frac{1}{2}\right) = 3\left(\frac{1}{2}\right)^2 - 2 = \frac{-5}{4}; \text{ thus } \frac{-5}{4} \text{ is paired with } \frac{1}{2}.$$

$$f\left(\frac{-2}{3}\right) = 3\left(\frac{-2}{3}\right)^2 - 2 = \frac{-2}{3}; \text{ thus } \frac{-2}{3} \text{ is paired with } \frac{-2}{3}.$$

The symbol $f(x)$ is read "f of x."

In another problem we may agree to let $f(x) = \dfrac{x^2 + 1}{x^2 - 1}$. Then throughout this problem whenever we refer to $f(x)$ we mean this expression. For this problem

$$f(2) = \frac{2^2 + 1}{2^2 - 1} = \frac{5}{3},$$

$$f(0) = \frac{0 + 1}{0 - 1} = -1,$$

$f(1)$ is meaningless.

We see that the range of the variable x for this function must not include the number 1.

We may use letters other than f to indicate a function, and the independent variable may be a letter other than x. Thus we may have:

$$f(x) = (x - 3)^2 + 1,$$

$$\Phi(s) = \frac{s + 2}{s - 2},$$

$$g(t) = t^2 + 2t - 3,$$

$$P(x) = 5x^3 + 4x^2 - 2x + 7,$$

$$Q(x) = 5x^2 + 9x + 7,$$

$$\theta(t) = t,$$

$$f(x) = 7,$$

$$f(x) = 0.$$

This notation for a function is of great use in mathematics. For instance, if we should want to make a statement about all polynomials (which, of course, are functions), we might start by saying, "If $P(x)$ is a polynomial, then $P(x) \cdots$"

EXERCISE 15.2.1

Find the indicated numbers:

1. $f(x) = 5x^2 + x - 4$; $f(1), f(0), f(-1), f(2), f(\frac{4}{5})$.
2. $F(x) = 5x^3 - 4x - 5x + 4$; $F(1), F(0), F(2), F(-1), F(\frac{4}{5})$.
3. $f(x) = \frac{2}{3}x + 2$; $f(3), f(-3), f(0), f(a), f(b)$.
4. $P(x) = 5x^3 + 4x^2 - 2x + 7$; $P(0)$, $P(1)$, $P(-1)$, $P(\frac{1}{2})$, $P(\frac{2}{3})$.
5. $g(t) = t^2 + 2t - 3$; $g(3), g(-3), g(1), g(x), g(x + 1)$.
6. $\Phi(x) = \dfrac{x + 1}{x - 1}$; $\Phi(2)$, $\Phi\left(\dfrac{a}{b}\right)$, $\Phi\left(\dfrac{a + 1}{a}\right)$, $\Phi(-1) + \Phi(\frac{1}{2})$.
7. $f(x) = 3x^2$; $[f(3) - f(2)] - [f(2) - f(1)]$, $[f(4) - f(3)] - [f(3) - f(2)]$, $[f(5) - f(4)] - [f(4) - f(3)]$.

8. Show that for the function of Problem 7:

$$[f(n + 2) - f(n + 1)] - [f(n + 1) - f(n)] = \text{a constant.}$$

9. Show that for $F(x) = 5x^2 - x + 2$:

$$[F(n + 2) - F(n + 1)] - [F(n + 1) - F(n)] = \text{a constant.}$$

10. Show that for $\Phi(x) = ax^2 + bx + c$:

$$[\Phi(n + 2) - \Phi(n + 1)] - [\Phi(n + 1) - \Phi(n)] = \text{a constant.}$$

15.3 Rectangular Coordinates

We have seen how we may label with a number each point on our number scale (Fig. 12.3.1). This may be looked upon as a geometric interpretation of our number system. To each number there corresponds a point, and to each point there corresponds a number. We say that we have a one-to-one correspondence between our numbers and the points of the number scale. This geometric interpretation of our number system becomes vivid when, instead of speaking of the difference between two numbers, we speak of the *distance* between two points. Thus the difference between 5 and -2 is 7, and the distance between the point corresponding to -2 and the point corresponding to 5 is 7 units.

The idea of giving a geometric interpretation to a *pair* of numbers gradually developed in the minds of mathematicians. Its final impetus was the publication of Descartes' *La Géométrie* in 1637.* This is generally considered the first book on analytic geometry, the study of the relation of algebra and geometry.

Suppose we choose two lines intersecting at right angles. On each of them we set up a number scale with the origin of each at the point of intersection. On the horizontal line we take positive numbers to the right and negative to the left, as we have previously done. On the vertical line we take positive numbers up and negative down. Instead of labeling each point on the horizontal line (or *axis of abscissas* or *x-axis* as it is called) with a number x, we label the point with the *number-pair* $(x, 0)$. Thus, where before we should have 2, we now have (2, 0). Where before we should have −3, we now have (−3, 0).

---

* Many of the ideas used by Descartes go back to others, even to ancient times. The idea of coordinates was known to the ancient Egyptians; the idea of the relation of algebra and geometry is contained in the Pythagorean theorem; Pierre de Fermat, a contemporary of Descartes, discussed the fundamental ideas in a letter written a year before the publication of Descartes' book.

Similarly we label each point on the vertical line (or *axis of ordinates* or *y-axis*) with the number-pair (0, y). This gives us a

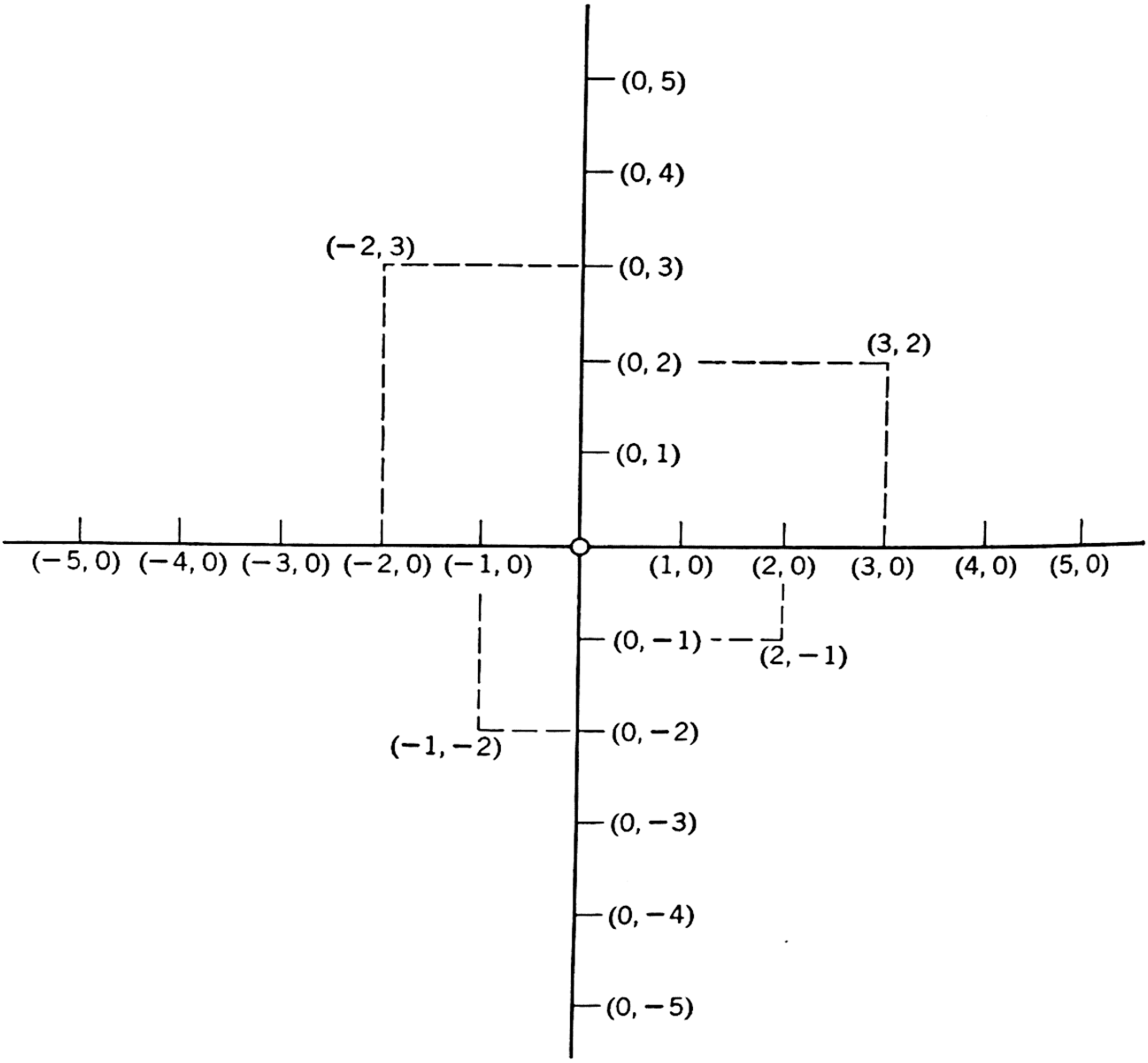


FIG. 15.3.1

method of associating each number-pair of the form (x, 0) or of the form (0, y) with one and only one point.

The number-pair (x, y), where neither x nor y is zero, is now associated with the point whose projection on the x-axis is the point labeled (x, 0) and whose projection on the y-axis is the point labeled (0, y). We have now set up a one-to-one correspondence between number-pairs and the points of the plane. This is called a two-dimensional coordinate system; this particular system is called the rectangular coordinate system or the Cartesian system in honor of Descartes.* We usually refer to the point with label (x, y) as "the point (x, y)" and call x its *abscissa*, y its *ordinate*, and x and y its *coordinates.*

In order to expedite the location of a point and to make its location more accurate, we like to use graph paper or cross-section paper. This is simply paper with horizontal and vertical lines drawn at convenient equal distances apart. However, you should also learn to locate points quickly and quite accurately without graph paper. When we are locating a point we say that we are "plotting the point."

Exercise 15.3.2

1. Plot the points $(2, 3)$, $(-3, 2)$, $(-2, -1)$, $(1, -2)$, $(0, 0)$, $(3, 0)$, $(0, 3)$, $(-5, 0)$, $(0, -5)$.

2. Plot points of the form $(x, 3x - 2)$ for $x = -3, -2, -1, 0, 1, 2$, and 3.

3. Plot points of the form $(x, -2x + 3)$ for $x = -3, -2, -1, 0, 1, 2, 3, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{3}, \frac{1}{3}, \frac{5}{4}, -\frac{5}{4}$.

4. Plot points of the form $(x, |x|)$ for $x = -3, -2, -1, 0, 1, 2$, and 3.

5. Plot points of the form (x, x^2) for $x = -3, -2, -1, 0, 1, 2, 3, -\frac{2}{3}, -\frac{1}{3}, \frac{1}{3}$, and $\frac{2}{3}$.

6. Plot points of the form $(x, 3x^2 - 2)$ for $x = -3, -2, -1, 0, 1, 2, 3, -\frac{2}{3}, -\frac{1}{3}, \frac{1}{3}, \frac{2}{3}$.

7. Plot points of the form $(x, 3[x])$ for $x = 0, \frac{1}{100}, \frac{1}{10}, \frac{9}{10}, \frac{99}{100}, 1, \frac{11}{10}, \frac{3}{2}, \frac{19}{10}, 2, \frac{21}{10}, \frac{29}{10}, 3, \frac{31}{10}, \frac{39}{10}$.

8. Plot points of the form (x, y) where $y = x^2$ for $x = -3, -2, -1, 0, 1, 2$.

15.4 Graphs of Functions

We have seen that a function establishes a correspondence or pairing of numbers. The function $f(x) = 3x^2 - 2$ pairs a number x with a number obtained by squaring x, multiplying the result by 3, and then subtracting 2. We saw that this function pairs 3 with 25, 1 with 1, 0 with -2, $\frac{1}{2}$ with $-\frac{5}{4}$, $-\frac{2}{3}$ with $-\frac{2}{3}$. It would be more convenient to write these number-pairs as (3, 25), (1, 1), (0, -2), ($\frac{1}{2}$, $-\frac{5}{4}$), ($-\frac{2}{3}$, $-\frac{2}{3}$). As soon as we do this, we see that we can think of these number-pairs as coordinates of points. Indeed, Problem 6 of the last exercise is the problem of plotting some points whose coordinates are of the form $(x, 3x^2 - 2)$. The collection of all points of this form and only this form is called the graph of the function $f(x) = 3x^2 - 2$.

---

* Other systems are used, in particular the polar system. For our present purposes the Cartesian system is most useful.

Polar coordinates also associate each point with a number-pair; in this case the number-pair is a distance and an angle. For further discussion look up "Polar Coordinates" in any analytic geometry book.

15.4.1 DEFINITION: The collection or set of all points of the form $(x, f(x))$ and only this form is called the *graph* of the function $f(x)$.

Frequently we want to give to a function a name shorter than $f(x)$. If there is no chance of being misunderstood, we may write $y = f(x)$. Then the graph of the function $y = f(x)$ is the collection of all points of the form (x, y) and only those points. Sometimes we say that we graph the equation $y = f(x)$.

Now offhand we should not expect the points obtained in this way to have any particular relationship to each other. We should not be surprised when we plot a large number of such points if we find them scattered helter-skelter all over the plane. In many cases our surprise is of just the opposite kind. We find the points related to each other in such a way that they seem to lie along geometric lines and curves. The points of Problems 2, 3, 4, 5, 6, and 8 are of this type.

When we seek to draw the graph of a function we should plot enough points so that we get a good idea of the shape of the graph and feel confident that we can tell just about where intermediate points lie.

EXERCISE 15.4.2

Graph the following functions:

1. $f(x) = 2x$.
2. $f(x) = 2x + 3$.
3. $f(x) = 2x - 3$.
4. $f(x) = 6x$.
5. $f(x) = x/2$.
6. $f(x) = -2x$.
7. $f(x) = -6x$.
8. $f(x) = -x/2$.
9. $f(x) = -2x + 4$.
10. $f(x) = -2x - 4$.
11. $f(x) = 5$.
12. $f(x) = -3$.
13. $f(x) = 0$.
14. $y = 3x + 2$.
15. $y = 5x + 2$.
16. $y = -x + 2$.
17. $y = -4x + 2$.
18. $y = 0$.

19. $y = x^2$.
20. $y = x^2 + 2$.
21. $y = x^2 - 2$.
22. $y = 2x^2$.
23. $y = x^2/2$.
24. $y = (x - 1)^2$.
25. $y = (x + 1)^2$.
26. $y = 2(x - 1)^2$.
27. $y = x^3$.
28. $y = (x - 1)^3$.

15.5 Linear Functions

When you plotted the graphs of the functions of Problems 1 through 10 of Exercise 15.4.2, you probably noticed that the points seemed to lie along straight lines. If you look at the form of these functions to see what is characteristic of them, you note that each of them is a polynomial of the first degree. It can be proved,* though we shall not prove it here, that

15.5.1 *If a function is a polynomial of the first degree, then its graph is a straight line.*

For this reason a polynomial of the first degree is frequently called a *linear* function.

From geometry we know that "two points determine a line." This means that as soon as we locate two points on a line the line is fixed. In plotting a linear function we need plot only two points and draw the line through them; however, it is advisable to plot at least three points as a check. The farther apart we plot the points, the more accurately our line will be located.

EXERCISE 15.5.2

Graph the following functions:

1. $y = 3x + 5$.
2. $y = 3x + 4$.
3. $y = 3x$.
4. $y = 3x - 2$.
5. $y = 3x - 5$.
6. $y = -3x + 5$.
7. $y = -3x + 4$.
8. $y = -3x$.
9. $y = -3x - 2$.
10. $y = -3x - 5$.
11. $y = -5x - 5$.
12. $y = -\dfrac{x}{2} - 5$.
13. $y = 5x - 5$.
14. $y = 1/x$.

What point appears to be common to the graphs of:

15. $y = 2x + 1$ and $y = -x + 7$.
16. $y = 4x$ and $y = -x + 10$.

---

* See any analytic geometry book.

17. $y = 7x$ and $y = 5x + 2$.
18. $y = 5x - 2$ and $y = 2x + 3$.
19. $y = x$ and $y = 3x + 4$.
20. $y = -x + 11$ and $y = 2x + 16$.

How do the values of these functions compare at the points where their graphs intersect?

16

Operations with Polynomials

16.1 Addition and Subtraction of Polynomials

We have frequently had problems which involve the addition or subtraction of polynomials. Indeed, each time that we were asked to collect like terms we were being asked to add or subtract polynomials.

EXAMPLES 16.1.1

1. $(3x^2 + 2x + 5) + (5x^2 - 4x - 7)$

$$= (3 + 5)x^2 + (2 - 4)x + (5 - 7)$$
$$= 8x^2 - 2x - 2.$$

2. $(7x^3 + 2x - 11) - (4x^3 - 5x^2 - 9)$

$$= (7x^3 + 2x - 11) + (-4x^3 + 5x^2 + 9)$$
$$= (7 - 4)x^3 + 5x^2 + 2x + (-11 + 9)$$
$$= 3x^3 + 5x^2 + 2x - 2.$$

Often it is convenient to write under one another the polynomials which are to be added or subtracted, with like terms under each other, and then add or subtract by columns.

3. To add $8x^4 + 5x^3 - 2x - 17$ to $9x^3 + 2x^2 + 2x + 11$, we write

$$\begin{array}{rrrrr} 8x^4 & +\ 5x^3 & & -\ 2x & -\ 17 \\ & 9x^3 & +\ 2x^2 & +\ 2x & +\ 11 \\ \hline 8x^4 & +\ 14x^3 & +\ 2x^2 & & -\ 6. \end{array}$$

We sometimes say in a case such as the above that "the second-degree term is missing" from the first polynomial. Or we might say that the "*coefficient of* x^2 *in this polynomial is zero*" and write $8x^4 + 5x^3 - 2x - 17 = 8x^4 + 5x^3 + 0x^2 - 2x - 17$.

4. To subtract $9x^4 + 3x^2 - 2x + 15$ from $10x^5 - 7x^3 + 8x^2 - 12$, we write

$$\begin{array}{rrrrrr}
10x^5 & & -\,7x^3 & +\,8x^2 & & -\,12 \\
& 9x^4 & & +\,3x^2 & -\,2x & +\,15 \\
\hline
10x^5 & -\,9x^4 & -\,7x^3 & +\,5x^2 & +\,2x & -\,27.
\end{array}$$

Here we have (mentally) changed the signs of the subtrahend and added. Often it is better to actually change the signs of the subtrahend as we write it down. This problem then would appear as the *addition* problem:

$$\begin{array}{rrrrrr}
10x^5 & & -\,7x^3 & +\,8x^2 & & -\,12 \\
& -\,9x^4 & & -\,3x^2 & +\,2x & -\,15 \\
\hline
10x^5 & -\,9x^4 & -\,7x^3 & +\,5x^2 & +\,2x & -\,27.
\end{array}$$

In operating with polynomials it is best, though not essential, that first each be written in standard form (page 143).

EXERCISE 16.1.2

1. Add $2x^2 - 7x + 3$ and $5x^2 + 11x - 7$.
2. Add $4x^3 + 6x^2 - 5x + 2$ and $3x^3 - 2x^2 + 7x + 2$.
3. Add $8x^3 + 3x - 5$ and $5 - 6x^2 + 2x^3$.
4. Add $8 + 4x^3 + 2x^5$ and $6x^4 + 3x^2 + 1$.
5. Add $-5x^3y + 2xy^3 - 3y^4$ and $4x^4 + 3x^3y - 2x^2y^2 + 7xy^3$.
6. Subtract $3x^2 - 4x + 11$ from $5x^2 + x - 4$.
7. Subtract $5x^4 + 11x^2 - 12$ from $4x^3 - 2x + 7$.
8. Subtract $8x^3 - 2x^2 + 4x + 1$ from $5x^3 + 2x^2 + 4x - 3$.
9. Subtract $x^3y - 2x^2y^2 + 4xy^3 + 5y^4$ from $x^4 - 3x^2y^2 - 4xy^3 + 5y^4$.
10. Subtract $x^3 - 2x + 5$ from the sum of $3x^3 + 2x^2 - 5$ and $x^2 + 4x + 5$.
11. Subtract $x + 3$ from the sum of $x^5 + 2$ and $x^4 + x^2 + 1$.
12. Subtract the sum of $x^2 + 2x - 3$ and $x^3 + 3x^2 + 5$ from $2x^3 + 3x - 5$.

13. Subtract $3x^5 - 4x^4 + 3x^3 + 2x - 5$ from the sum of $x^4 + 5x^3 + 3x^2 - 7$ and $4x^5 + 2x^4 - 3x + 11$.

14. Subtract the sum of $x^3 + 2x - 11$ and $2x^2 + x - 1$ from the sum of $3x^4 + x^3 - 2x^2 + 1$ and $2x^3 + x^2 - 1$.

15. Add

$$\begin{array}{rrrrrr}
 & & -\,18x^3 & +\,27x^2 & -\,63x & -\,18 \\
 & 8x^4 & -\,12x^3 & +\,28x^2 & +\,8x & \\
10x^5 & -\,15x^4 & +\,35x^3 & +\,10x^2 & & \\
\hline
\end{array}$$

16. Add

$$\begin{array}{rrrrrr}
15x^5 & +\,9x^4 & +\,12x^3 & -\,21x^2 & & \\
 & & -\,10x^3 & -\,6x^2 & -\,8x & +\,14 \\
\hline
\end{array}$$

17. Add

$$\begin{array}{rrrrrrr}
2x^7 & +\,5x^5 & & -\,7x^3 & & & \\
 & -\,2x^5 & & -\,5x^3 & & +\,7x & \\
 & & 2x^4 & & +\,5x^2 & & -\,7 \\
\hline
\end{array}$$

18. Add

$$\begin{array}{rrrrrrrr}
 & & & -\,7x^4 & & +\,21x^2 & -\,14x & +\,21 \\
 & & 2x^5 & & -\,6x^3 & +\,4x^2 & -\,6x & \\
3x^8 & -\,9x^6 & +\,6x^5 & -\,9x^4 & & & & \\
\hline
\end{array}$$

16.2 Multiplication of Polynomials

We have had numerous examples in multiplication of polynomials. Each such product is found by use of the distributive law. In certain type cases you were expected to write the product at sight (13.2.1).

When we have a problem of multiplying two polynomials which is not one of these types it may be difficult to write the product at sight. We therefore devise a system to obtain such products.

Consider the product $(2x^3 - 3x^2 + 7x + 2)(5x^2 + 4x - 9)$. By the distributive law this equals

$$\begin{aligned}
&5x^2(2x^3 - 3x^2 + 7x + 2) + 4x(2x^3 - 3x^2 + 7x + 2) \\
&\qquad\qquad\qquad\qquad - 9(2x^3 - 3x^2 + 7x + 2).
\end{aligned}$$

We now perform each of the indicated multiplications in order from left to right (or from right to left) and collect like terms. After we multiply, we have the problem of adding the three polynomials:

$$\begin{array}{rrrrrr}
10x^5 & -\ 15x^4 & +\ 35x^3 & +\ 10x^2 & & \\
 & 8x^4 & -\ 12x^3 & +\ 28x^2 & +\ 8x & \\
 & & -\ 18x^3 & +\ 27x^2 & -\ 63x & -\ 18 \\
\hline
\end{array}$$

or, if we wrote them in reverse order (which is what we should obtain if we did the multiplications in order from right to left), we should have Problem 15 of Exercise 16.1.2.

This example illustrates the system we use to multiply two polynomials whose product we cannot find at sight. The next example illustrates the form we use.

Multiply $2x^3 + 3x - 4$ by $5x - 3$:

$$\begin{array}{rrrrr}
 & 2x^3 & +\ 3x & -\ 4 & \\
 & & 5x & -\ 3 & \\
\hline
10x^4 & & +\ 15x^2 & -\ 20x & \\
 & -\ 6x^3 & & -\ 9x & +\ 12 \\
\hline
10x^4 & -\ 6x^3 & +\ 15x^2 & -\ 29x & +\ 12
\end{array}$$

EXERCISE 16.2.1

Multiply:

1. $(x^2 + 2x + 1)(x + 1)$.
2. $(3x^2 + 4x - 5)(x^2 - x + 3)$.
3. $(5x^3 + 3x^2 + 4x - 7)(3x^2 - 2)$.
4. $(2x^4 + 5x^2 - 7)(3x^2 - 2)$.
5. $(4x^5 - 3x^3 + 5x)(x^4 + 2x^2 + 7)$.
6. $(3x^4 + 2x^3 + 4x + 7)(2x^3 + 4x + 5)$.

7. $(x + 2)^3$. 8. $(2x + 3)^3$. 9. $(3x - 2)^3$. 10. $(x + 1)^4$.
11. $(x + a)^3$. 12. $(x - a)^3$. 13. $(x + a)^4$. 14. $(x - a)^4$.

Show that $P(x) = (x - a)Q(x) + R$ for each of the following. Verify your work for the given particular values of x.

15. $P(x) = x^3 - 2x^2 + 2x - 1$,
$Q(x) = x^2 - x + 1$;
$a = 1$, $R = 0$,
$x = 1, 2, -1$, and -2.

16. $P(x) = x^3 + 2x^2 + 2x + 1$,
$Q(x) = x^2 + x + 1$;
$a = -1$, $R = 0$,
$x = 1, 2, -1$, and -2.

17. $P(x) = x^4 - x^3 + x^2 - 1$,
$Q(x) = x^3 + x + 1$;
$a = 1$, $R = 0$,
$x = 1, 2, -1, -2$.

18. $P(x) = x^5 + 2x^4 + 2x^3 + 2x^2 + 2x + 1$,
$Q(x) = x^4 + x^3 + x^2 + x + 1$;
$a = -1$, $R = 0$,
$x = 1, 2, -1, -2$.

19. $P(x) = x^3 - 2x + 2$,
$Q(x) = x^2 + 2x + 2$;
$a = 2$, $R = 6$,
$x = 1, 2, -1, -2$.

20. $P(x) = x^5 + 2x^4 + 2x^3 + 3x^2 + x - 11$,
$Q(x) = x^4 + 2x^2 - x + 3$;
$a = -2$, $R = -17$,
$x = 1, 2, -1, -2$.

21. $P(x) = x^3 + 4x^2 + 6x + 7$,
$Q(x) = x^2 + 2x + 2$;
$a = -2$, $R = 3$,
$x = 10$.

22. If $P(x) = 3x^5 + 3x^4 + 7x^3 + 9x^2 + 5x + 8$,
$Q(x) = 3x^3 + 3x^2 + x + 3$,
$D(x) = x^2 + 2$,
$R(x) = 3x + 2$,

show that $P(x) = D(x) \cdot Q(x) + R(x)$. Verify for $x = 10$.

16.3 Division of Polynomials

An oil tank contains 2000 gallons of oil. It is connected to an oil burner which burns 27 gallons of oil a day. How long will the oil last?

This problem may be solved by repeated subtraction. At the end of the first day the number of gallons left will be $2000 - 27 = 1973$; at the end of the second day there will be $1973 - 27 = 1946$, etc., until a remainder of less than 27 is obtained. The number of times the subtraction may be performed corresponds to the number of days the oil will last. We may note that, if we know no arithmetic other than subtraction and counting, we could solve the problem in this way, by counting how many times we can remove (subtract) 27 gallons.

However, we do know more arithmetic, and we can solve the problem by division. We divide 2000 by 27, obtaining the quotient 74 and a remainder 2. The quotient 74 represents the number of days that the oil will last (that is, the number of times we can subtract 27) and the remainder 2 represents the amount of oil left over. We have

$$2000 - 27 \cdot 74 = 2$$

or

$$2000 = 27 \cdot 74 + 2.$$

We have previously defined division of integers (10.1.1). At that time we stated that to divide an integer p by an integer d (not zero) meant to find a number q such that $p = d \cdot q$. Frequently the number q turns out to be not an integer, though it is always a rational number; thus $7 \div 2 = 7/2$.

The new concept of division, illustrated by our oil-tank problem, involves positive integers only.

16.3.1 Definition: To divide a positive integer p, the dividend, by a positive integer d, the divisor, $(d \leqq p)$ means to find a positive integer q, the quotient, and a non-negative integer r, the remainder, such that

$$p = d \cdot q + r \text{ and } 0 \leqq r < d.$$

16.3.2 Definition: If the remainder r is zero, then d is a divisor of p. (Compare with 10.1.3.)

We emphasize that we are asking, "How many times can we subtract d from p and, when we have done this as often as possible, how much is left over?" This is perhaps seen more clearly when we write

$$p - dq = r, \quad 0 \leqq r < d.$$

EXAMPLES 16.3.3

1. To divide 37 by 5 means to find a positive integer q and a non-negative integer r such that $37 = 5 \cdot q + r$ or such that $37 - 5 \cdot q = r$. We see that $q = 7$ and $r = 2$ are the quotient and remainder, respectively, since

$$37 = 5 \cdot 7 + 2$$

or

$$37 - 5 \cdot 7 = 2.$$

We can subtract 5 from 37 seven times and will then have a remainder 2.

2. To divide 123 by 5 means to find a positive integer q and a non-negative integer r such that $123 = 5 \cdot q + r$ or such that $123 - 5 \cdot q = r$.

We make our work easier here, if we first ask how many times we can subtract $10 \cdot 5$, that is, first divide by 50. We have

$$123 - (10 \cdot 5) \cdot 2 = 23$$

or

$$123 - 20 \cdot 5 \qquad = 23.$$

But

$$23 = 4 \cdot 5 + 3$$

so that we finally have

$$\begin{aligned} 123 - 20 \cdot 5 &= 4 \cdot 5 + 3 \\ (123 - 20 \cdot 5) - 4 \cdot 5 &= 3 \\ 123 - (20 + 4) \cdot 5 &= 3 \\ 123 &= 5 \cdot 24 + 3, \end{aligned}$$

and we see the quotient is 24 and the remainder is 3.

3. To divide 675 by 21:

$$675 = (10\cdot 21)\cdot 3 + 45$$

$$675 = 210\cdot 3 + 45$$

$$675 - 21\cdot 30 = 45 \quad \text{(45 is left over after we subtract 21 thirty times)}$$

$$45 = 21\cdot 2 + 3$$

$$675 - 21\cdot 30 = 21\cdot 2 + 3$$

$$675 - 21(30 + 2) = 3$$

$$675 = 21\cdot 32 + 3$$

so that the quotient obtained when 675 is divided by 21 is 32 and the remainder is 3.

These examples illustrate the method we called "long division" in arithmetic. The last example in the form that we learned looks like

$$\begin{array}{rll}
 & \overline{\;32 = q\;} = 30 + 2 & \\
d = 21\big) & 675 = p & \\
 & 630 = 30\cdot d & \text{(subtract from above)} \\
\cline{2-2}
 & 45 = p - 30\cdot d & \\
 & 42 = 2d & \text{(subtract from above)} \\
\cline{2-2}
 & 3 = r = p - 32d. &
\end{array}$$

16.3.4 **Definition:** To divide a polynomial $P(x)$ by a polynomial $D(x)$ means to find a polynomial $Q(x)$ and a remainder $R(x)$ such that

$$P(x) = D(x)\cdot Q(x) + R(x)$$

and $R(x)$ is a polynomial of lower degree than $D(x)$ or a constant.

16.3.5 **Definition:** If $R(x) = 0$, then $D(x)$ is a divisor of or factor of $P(x)$.

If we write the equality defining division in the form

$$P(x) - D(x)\cdot Q(x) = R(x),$$

we see that this also is repeated subtraction. We subtract from

$P(x)$ as great a multiple of $D(x)$ as we can. The work can be arranged in a manner similar to long division.

EXAMPLES 16.3.6

1. To divide $x^3 + 3x^2 - 4x - 14$ by $x + 2$:

$$\begin{array}{rrcl}
 & x^2 + x - 6 & = & Q(x) \\
\cline{2-2}
D(x) = x + 2) & x^3 + 3x^2 - 4x - 14 & = & P(x) \\
 & x^3 + 2x^2 \phantom{{} - 4x - 14} & = & x^2 \cdot D(x) \\
\cline{2-2}
 & x^2 - 4x - 14 & = & P(x) - x^2 \cdot D(x) \\
 & x^2 + 2x \phantom{{} - 14} & = & x \cdot D(x) \\
\cline{2-2}
 & -6x - 14 & = & P(x) - x^2 \cdot D(x) - x \cdot D(x) \\
 & -6x - 12 & = & -6 \cdot D(x) \\
\cline{2-2}
 & -2 & = & P(x) - x^2 \cdot D(x) - x \cdot D(x) \\
 & & & \quad - (-6)D(x) \\
 & & = & P(x) - (x^2 + x - 6)D(x) \\
 & & = & R(x)
\end{array}$$

so that

$$x^3 + 3x^2 - 4x - 14 = (x + 2)(x^2 + x - 6) - 2.$$

Note that we do not require that the remainder be positive but only that it be of lower degree than $D(x)$ or a constant.

2. To divide $6x^2 + 7x + 5$ by $2x + 1$:

$$\begin{array}{rr}
 & 3x + 2 \phantom{{} + 5} \\
\cline{2-2}
2x + 1) & 6x^2 + 7x + 5 \\
 & 6x^2 + 3x \phantom{{} + 5} \\
\cline{2-2}
 & 4x + 5 \\
 & 4x + 2 \\
\cline{2-2}
 & 3
\end{array}$$

$$6x^2 + 7x + 5 = (2x + 1)(3x + 2) + 3.$$

Compare this with Example 3 of Examples 16.3.3.

EXERCISE 16.3.7

Find $Q(x)$ and $R(x)$ and write in the form $P(x) = D(x)Q(x) + R(x)$. Check your results.

1. $P(x) = 3x^2 + 7x + 5$, $D(x) = x + 2$.
2. $P(x) = 4x^3 + 9x^2 + 5x + 1$, $D(x) = x + 2$.
3. $P(x) = 5x^3 - 16x^2 + 2x + 7$, $D(x) = x - 3$.
4. $P(x) = 12x^3 - 4x^2 - 29x + 6$, $D(x) = 2x + 3$.
5. $P(x) = 14x^4 - 9x^3 - 20x^2 + 9x - 9$, $D(x) = 2x - 3$.
6. $P(x) = 12x^4 + 13x^3 - 4x^2 - 6x - 11$, $D(x) = 3x + 4$.
7. $P(x) = 5x^4 + 3x^3 + 2x^2 + x + 3$, $D(x) = x^2 + x + 1$.
8. $P(x) = 5x^4 + 3x^3 + 2x^2 - 3x - 1$, $D(x) = x^2 + x + 1$.
9. $P(x) = 3x^5 + 3x^4 + 7x^3 + 9x^2 + 5x + 8$, $D(x) = x^2 + 2$.
10. $P(x) = 8x^4 + 2x^2 + 7$, $D(x) = 2x + 3$.
11. $P(x) = x^3 - 6x + 5$, $D(x) = x^2 - 3$.
12. $P(x) = 12x^4 + 2x^2 - 11$, $D(x) = 3x^2 + 2$.
13. $P(x) = 3x^5 - 4x^3 + x$, $D(x) = x^2 + 2x - 1$.
14. $P(x) = 2x^5 - 2x^3 + 15x^2 - 6x + 9$, $D(x) = x^2 - 2x + 3$.
15. Show that $x - 1$ is a factor of $x^3 - 2x^2 - x + 2$, and find all the solutions of the equation $x^3 - 2x^2 - x + 2 = 0$.
16. Show that $3x - 2$ is a factor of $3x^3 + 19x^2 + 16x - 20$, and find all the solutions of $3x^3 + 19x^2 + 16x - 20 = 0$.
17. Show that $5x - 7$ is a factor of $P(x) = 20x^3 - 28x^2 - 45x + 63$, and find all the solutions of $P(x) = 0$.
18. Show that $x^2 + 2x - 15$ is a factor of $P(x) = x^6 + 2x^5 - 20x^4 - 10x^3 + 79x^2 + 8x - 60$, and find all the solutions of $P(x) = 0$.

16.4 Decimal Polynomials and Decimal Notation

If we consider the meaning of a number symbol such as 352 written in our notation (which you probably know is called the *decimal notation*) we realize that it means:

$$352 = 3 \cdot 100 + 5 \cdot 10 + 2 = 3 \cdot 10^2 + 5 \cdot 10 + 2.$$

If we replace the 10 by x (it is a happy accident that X is the symbol for 10 in the Roman notation), we obtain $352 = 3x^2 +$

$5x + 2$, a polynomial in x. We note that any integer written in our notation may be considered such a polynomial if $x = 10$.*

16.4.1 DEFINITION: A *decimal polynomial* is a polynomial in $x = 10$ whose coefficients are each one of the numbers 0, 1, 2, 3, 4, 5, 6, 7, 8, or 9, and no others. These coefficients are called the *digits*.

EXAMPLES 16.4.2

If $x = 10$:

1. $x^2 + 2x + 1$ is a decimal polynomial.
2. $5x^3 + 2x^2 + 3x + 7$ is a decimal polynomial.
3. $6x^2 + 0x + 5 = 6x^2 + 5$ is a decimal polynomial.
4. $9x^5 + 9x^3 + 9x$ is a decimal polynomial.
5. $3x^2 + 2x - 1$ is *not* a decimal polynomial.
6. 532, in the decimal notation, as a decimal polynomial is $5x^2 + 3x + 2$.
7. 603, in the decimal notation, as a decimal polynomial is $6x^2 + 3$.
8. $3x^2 + 2x + 5$, as a decimal polynomial, in the decimal notation is 325.
9. $5x^3 + 2x$, as a decimal polynomial, in the decimal notation is 5020.

We see that our decimal notation is simply a shorthand way of representing a decimal polynomial. Just as the polynomials

$$3x^2 + 5x + 2,\ 3x^2 + 2x + 5,$$

$$2x^2 + 5x + 3,\ 2x^2 + 3x + 5,$$

$$5x^2 + 2x + 3,\ 5x^2 + 3x + 2$$

are each different from the others, so

$$352,\ 325,\ 253,\ 235,\ 523,\ 532$$

---

* It can be shown that any integer may be considered as a polynomial in r, where r is any natural number not less than 2, the coefficients of the polynomial each being one of the numbers $0, 1, 2, 3, \cdots, (r - 1)$. Thus 27 (in decimal notation) may be written as $r^2 + 2r + 3$ if $r = 4$ and as $r^4 + r^3 + 0 \cdot r^2 + r + 1$ if $r = 2$. For a further discussion see Richardson: *Fundamentals of Mathematics*, page 165 et seq.

are each different numbers. They contain the same digits but the *positions* of the digits are different. The meaning of a digit depends not only on the digit but also on its position. The 2 in 352 means 2 but in 235 it means 200. The position of the digit determines the power of 10 which it multiplies. The constant term in a decimal polynomial is called the *units digit,* the coefficient of x is called the *tens digit,* the coefficient of x^2 is called the *hundreds digit,* etc. For $7x^3 + 3x^2 + 2x + 5$, 5 is the units digit, 2 is the tens digit, 3 is the hundreds digit, and 7 is the thousands digit. This same statement applies to the decimal form 7325.

When a power of 10 is "missing," as is the first power in $8x^2 + 9$, this simply means that the coefficient of that power (or the tens digit) is zero. We could write it $8x^2 + 0 \cdot x + 9$, if we desired. When we write our number in decimal notation, we *must* write the zero, as 809. Otherwise we should have the number 89.

Operations with decimal polynomials follow the same rules as those which hold for ordinary polynomials, except that we must always keep in mind that no coefficient can be other than one of the digits and that x represents ten. If we obtain, through some operation, a coefficient which, in calculation with ordinary polynomials, would be ten or more or negative we must change it by using these two restrictions.

Examples 16.4.3

1. To add $2x + 6$ and $5x + 5$ we write

$$\begin{array}{rl} 2x + 6 & \\ 5x + 5 & \\ \hline 7x + 11 & = 7x + (10 + 1) \\ & = 7x + x + 1 \\ & = x(7 + 1) + 1 \\ & = 8x + 1. \end{array}$$

In decimal notation this would be

$$\begin{array}{r} 26 \\ 55 \\ \hline 81. \end{array}$$

Note that by considering decimal polynomials we can learn to understand the process of "carrying" which we perhaps learned by rote in arithmetic.

2. Add

$$\begin{array}{r}
6x + 7 \\
7x + 5 \\
\hline
13x + 12
\end{array}
\begin{array}{l}
\\
\\
\\
= (10 + 3)x + (10 + 2) \\
= (x + 3)x + (x + 2) \\
= x^2 + 3x + x + 2 \\
= x^2 + 4x + 2.
\end{array}$$

Write this in decimal notation. Pay particular attention to the "carrying" operation.

3. Add

$$\begin{array}{r}
5x^2 \phantom{{}+6x} + 2 \\
3x^3 + 5x^2 + 6x + 9 \\
\hline
3x^3 + 10x^2 + 6x + 11
\end{array}
\begin{array}{l}
\\
\\
\\
= 3x^3 + x \cdot x^2 + 6x + (x + 1) \\
= 3x^3 + x^3 + 6x + x + 1 \\
= 4x^3 + 7x + 1.
\end{array}$$

4. Subtract

$$\begin{array}{r}
6x + 3 \\
4x + 5 \\
\hline
\end{array}$$

We cannot subtract 5 from 3 and obtain a digit and so we write

$$6x + 3 = 5x + x + 3 = 5x + 10 + 3 = \begin{array}{r}
5x + 13 \\
4x + 5 \\
\hline
x + 8.
\end{array}$$

Which step in our work corresponds to "borrowing"?

5. Subtract

$$\begin{array}{r}
3x^2 + 2x + 4 \\
x^2 + 5x + 6 \\
\hline
\end{array}$$

Since we cannot subtract 6 from 4 (and obtain a digit), we write

$$3x^2 + 2x + 4 = 3x^2 + (x + x) + 4 = 3x^2 + x + 14.$$

But we now should have to subtract $5x$ from x, and we cannot do this and obtain a digit as coefficient. We write

$$\begin{aligned} 3x^2 + x + 14 &= 2x^2 + x^2 + x + 14 \\ &= 2x^2 + 10x + x + 14 \\ &= 2x^2 + 11x + 14, \end{aligned}$$

and we now have the subtraction problem

$$\begin{array}{r} 2x^2 + 11x + 14 \\ x^2 + 5x + 6 \\ \hline x^2 + 6x + 8. \end{array}$$

EXERCISE 16.4.4

Perform the indicated operation; each polynomial is a decimal polynomial:

1. $(3x + 7) + (2x + 8)$.
2. $(5x + 2) + (6x + 5)$.
3. $(7x + 6) + (4x + 6)$.
4. $(3x^2 + 4x + 2) + (5x^2 + 7x + 5)$.
5. $(5x^2 + 3x + 9) + (7x^2 + 8x + 1)$.
6. $(3x^2 + 9x + 9) + (6x^2 + x + 1)$.
7. $(8x + 2) - (6x + 7)$.
8. $(9x + 5) - (8x + 7)$.
9. $(3x^2 + 2x + 5) - (x^2 + 8x + 2)$.
10. $(5x^2 + 3x + 7) - (3x^2 + 4x + 8)$.
11. $(4x^2 + 5) - (3x^2 + 8)$.
12. $(2x + 3) \cdot (3x + 2)$.
13. $(4x + 6) \cdot (5x + 3)$.
14. $(3x + 4)^2$.
15. $(x + 4)^3$.
16. $(x^2 + 6x + 9) \div (x + 3)$.
17. $(2x^2 + 9x + 9) \div (x + 3)$.
18. $(3x^2 + x + 2) \div (x + 3)$.
19. $(6x^2 + 2x + 1) \div (2x + 3)$.

17

Integers as Exponents

17.1 Negative Integers and Zero as Exponents

We have defined a^n to mean the product of n a's. Of course, this definition makes sense only if n is a positive integer. It would be meaningless to say that a symbol such as $a^{1/2}$ meant "the product of ½ a's" or that a^{-2} meant "the product of -2 a's" (i.e., the ghosts of two departed a's). These symbols, $a^{1/2}$ and a^{-2}, and also, say, a^0, are at the moment meaningless. However, it has been found convenient to give meaning to such symbols. Whatever the meaning we assign to them, we shall require that these new kinds of exponents follow the same rules as those followed by positive integral exponents. We recall that these rules are:

1. $a^m \cdot a^n = a^{m+n}$.

2. $(a^m)^n = a^{m \cdot n}$.

3. $(a \cdot b)^m = a^m \cdot b^m$.

4. $\left(\dfrac{a}{b}\right)^m = \dfrac{a^m}{b^m}$.

5a. $\dfrac{a^m}{a^n} = a^{m-n}$ if $m > n$.

5b. $\dfrac{a^m}{a^n} = \dfrac{1}{a^{n-m}}$ if $m < n$.

5c. $\dfrac{a^m}{a^n} = 1$ if $m = n$.

In each of these rules it is assumed that there is no zero in the denominator.

Each of these rules was proved correct from the definition of a^n, n a positive integer. We called these rules the laws of exponents.

Rule 5 seems cumbersome with its three different cases. Suppose that we try to consolidate these three cases. We see how this may be done by considering the result of applying Rule 5a to, say, a^3/a^5. This gives us

$$\frac{a^3}{a^5} = a^{3-5} = a^{-2}.$$

Since by Rule 5b $a^3/a^5 = 1/a^2$, if there are to be no inconsistencies, we must have $a^{-2} = 1/a^2$. This suggests the

17.1.1 Definition: If n is an integer different from zero and $a \neq 0$, then $a^{-n} = 1/a^n$ or $a^n \cdot a^{-n} = 1$ or $a^n = 1/a^{-n}$.

This definition states that, if $a \neq 0$, then a^n and a^{-n} are reciprocals.

Similarly, if we apply Rule 5a to a/a, we obtain $a/a = a^{1-1} = a^0$; but $a/a = 1$. We therefore formulate the

17.1.2 Definition: If $a \neq 0$, then $a^0 = 1$.

We may now replace the three cases 5a, 5b, 5c by the single rule:

5. $\dfrac{a^m}{a^n} = a^{m-n} = \dfrac{1}{a^{n-m}}.$

It can be shown that all the laws of exponents, 1, 2, 3, 4, and the new 5 hold for all integral exponents.

Examples 17.1.3

1. $a^5 \cdot a^{-2} = a^{5-2} = a^3.$
2. $(x^7)^{-2} = x^{-14} = \dfrac{1}{x^{14}}.$
3. $(c^{-3})^{-2} = c^6$ or $(c^{-3})^{-2} = \dfrac{1}{(c^{-3})^2} = \dfrac{1}{c^{-6}} = c^6.$
4. $(p^{-4})^3 = p^{-12} = \dfrac{1}{p^{12}}$ or $(p^{-4})^3 = \left(\dfrac{1}{p^4}\right)^3 = \dfrac{1}{p^{12}}.$
5. $(a \cdot b)^{-3} = \dfrac{1}{(ab)^3} = \dfrac{1}{a^3b^3}$ or $(a \cdot b)^{-3} = a^{-3} \cdot b^{-3} = \dfrac{1}{a^3} \cdot \dfrac{1}{b^3}$ $= \dfrac{1}{a^3b^3}.$

6. $\left(\dfrac{x}{y}\right)^{-2} = \dfrac{x^{-2}}{y^{-2}} = \dfrac{x^{-2}}{1} \cdot \dfrac{1}{y^{-2}} = \dfrac{1}{x^2} \cdot \dfrac{y^2}{1} = \dfrac{y^2}{x^2} = \left(\dfrac{y}{x}\right)^2$ or

$$\left(\frac{x}{y}\right)^{-2} = \frac{1}{\left(\dfrac{x}{y}\right)^2} = \frac{1}{\dfrac{x^2}{y^2}} = \frac{y^2}{x^2} = \left(\frac{y}{x}\right)^2.$$

7. $\dfrac{p^5}{p^{-3}} = p^{5-(-3)} = p^8$ or $\dfrac{p^5}{p^{-3}} = p^5 \cdot p^3 = p^8.$

8. $\dfrac{s^{-7}}{s^{-3}} = s^{-7-(-3)} = s^{-4} = \dfrac{1}{s^4}$ or $\dfrac{s^{-7}}{s^{-3}} = \dfrac{s^3}{s^7} = \dfrac{1}{s^4}.$

9. $(137)^0 = 1.$

10. $\dfrac{1}{a^{-1} + b^{-1}} = \dfrac{1}{\dfrac{1}{a} + \dfrac{1}{b}} = \dfrac{ab}{b + a}.$

EXERCISE 17.1.4

Write each of the following with positive exponents and in lowest terms:

1. $3x^{-2}$.
2. $(3x)^{-2}$.
3. $5x^0$.
4. $(5x)^0$.
5. $(2x^{-1})(2x)^{-1}$.
6. $(-3x^{-2})^3$.
7. $(-3x^2)^{-3}$.
8. $a^{-3}b^4$.
9. $a^{-7} \cdot a^4$.
10. $-3xy^{-2}z^{-3}$.
11. $-3(xy)^{-2}z^{-3}$.
12. $(-3xy)^{-2} \cdot xy^2$.
13. $2a^{-3}(2a^{-2})^2$.
14. $2a^{-3}(2a^2)^{-2}$.
15. $a^{-1} + b^{-1}$.
16. $(a + b)^{-1}$.
17. $a(a^{-1} + b^{-1})$.
18. $\dfrac{a^2b^2(a^{-1} - b^{-1})}{a^2b^2(a^{-2} - b^{-2})}$.
19. $(ab)^{-1} + ab^{-1}$.
20. $\dfrac{a^{-1} + b^{-1}}{a^2 - b^2}$.
21. $\dfrac{a - b}{a^{-1} - b^{-1}}$.
22. $\dfrac{a^{-1} + b^{-1} + c^{-1}}{(abc)^{-1}}$.
23. $\dfrac{1}{x^{-2} - y^{-2}}$.
24. $\dfrac{e^x + e^{-x}}{2}$.
25. $\dfrac{e^x + e^{-x}}{e^x - e^{-x}}$.

26. $(r^{-1} + s^{-1})^{-1}$.

27. $3 \cdot 10^{-4}$.

28. $(2 \cdot 10^{-2})(4 \cdot 10^{3})$.

29. $(3 \cdot 10^{6})(5 \cdot 10^{-7})(2 \cdot 10^{4})$.

30. $\dfrac{(2 \cdot 10^{0})(15 \cdot 10)^{0}}{4 \cdot 10^{-3}}$.

31. $\dfrac{(7 \cdot 10^{-3})(2 \cdot 10^{10})}{8 \cdot 10^{6}}$.

We defined a polynomial as an integral rational expression. This means that no variables may appear in a denominator (13.4). If we write $3x^{-1} + 2x^{-2} + 5x^{-3}$, this looks like a polynomial but it is not a polynomial since x^{-1}, for instance, is simply another way of writing $1/x$. Since we know about negative exponents, we had better revise our definition of polynomial.

17.1.5 Definition: A polynomial in x of degree n (where n is a positive integer) is an expression of the form

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n$$

in which the a's are constants and $a_0 \neq 0$.

17.2 Decimal Fractions

Consideration of a decimal fraction, such as 0.325, leads us to see that this means

$$\begin{aligned}\tfrac{325}{1000} &= \tfrac{300}{1000} + \tfrac{20}{1000} + \tfrac{5}{1000}\\ &= \tfrac{3}{10} + \tfrac{2}{100} + \tfrac{5}{1000}\\ &= 3 \cdot 10^{-1} + 2 \cdot 10^{-2} + 5 \cdot 10^{-3}.\end{aligned}$$

(The 0 before the decimal point in 0.325 is merely to emphasize the location of the decimal point and to make sure that it does not get lost.)

If we should replace 10 by x (as we did for integers to obtain decimal polynomials), we have the expression $3x^{-1} + 2x^{-2} + 5x^{-3}$. We can use such expressions, in which the coefficients are, as with decimal polynomials, restricted to the digits, to help us understand operations with decimal fractions.

EXAMPLES 17.2.1

1. Add 0.325 to 0.447. We may write

$$\begin{array}{l}3x^{-1} + 2x^{-2} + 5x^{-3} \\ 4x^{-1} + 4x^{-2} + 7x^{-3} \\ \hline 7x^{-1} + 6x^{-2} + 12x^{-3}\end{array}$$

$$\begin{aligned}7x^{-1} + 6x^{-2} + 12x^{-3} &= 7x^{-1} + 6x^{-2} + (x + 2)x^{-3} \\ &= 7x^{-1} + 6x^{-2} + x^{-2} + 2x^{-3} \\ &= 7x^{-1} + 7x^{-2} + 2x^{-3}.\end{aligned}$$

2. Subtract 0.437 from 0.675. We write

$$(6x^{-1} + 7x^{-2} + 5x^{-3}) - (4x^{-1} + 3x^{-2} + 7x^{-3}).$$

Since we cannot subtract 7 from 5 and obtain a digit, we write

$$\begin{aligned}6x^{-1} + 7x^{-2} + 5x^{-3} &= 6x^{-1} + (6x^{-2} + x^{-2}) + 5x^{-3} \\ &= 6x^{-1} + (6x^{-2} + x \cdot x^{-3}) + 5x^{-3} \\ &= 6x^{-1} + 6x^{-2} + 10x^{-3} + 5x^{-3} \\ &= 6x^{-1} + 6x^{-2} + 15x^{-3}\end{aligned}$$

and the subtraction may now be performed without any further "borrowing."

EXERCISE 17.2.2

Analyze the following as was done in the above examples:

1. 0.436 + 0.457.
2. 0.582 + 0.376.
3. 0.276 + 0.385.
4. 12.6 + 23.7.
5. 37.65 + 78.55.
6. 3.545 + 12.706.
7. 0.763 − 0.348.
8. 2.71 − 1.98.
9. 52.375 − 36.287.
10. 0.0135 − 0.0027.
11. (0.23)(0.14).
12. (6.24)(0.38).
13. (7.43) ÷ (2.3).
14. (14.77) ÷ (0.123).

Write as decimal fractions:

15. $\frac{1}{2}$.
16. $\frac{31}{50}$.
17. $\frac{23}{40}$.
18. $\frac{1}{3}$.
19. $\frac{2}{7}$.
20. $\frac{2}{23}$.

17.3 Repeating or Periodic Decimals

Since $\frac{1}{2} = \frac{5}{10}$ we may write $\frac{1}{2} = 0.5$; since $\frac{31}{50} = \frac{62}{100}$ we may write $\frac{31}{50} = 0.62$; since $\frac{23}{40} = \frac{575}{1000}$ we may write $\frac{23}{40} = 0.575$. In each of these examples the denominator of the given fraction is a divisor of a power of 10. The resulting decimal fraction is then a "terminating decimal"; there is a last non-zero digit. (Of course, we may, if we wish, write $0.575 = 0.5750000 \cdots$.)

But, if we consider the fractions $\frac{1}{3}$ or $\frac{2}{7}$, there are no powers of 10 of which the denominators are divisors. If we write the terminating decimals

$$\begin{aligned} S_1 &= 0.3, \\ S_2 &= 0.33, \\ S_3 &= 0.333, \\ S_4 &= 0.3333, \text{ etc.,} \end{aligned}$$

we see that

$$\begin{aligned} S_1 &= \frac{3}{10} = \frac{1}{3} - \frac{1}{30} = \frac{1}{3} - \frac{1}{3\cdot(10)}, \\ S_2 &= \frac{33}{100} = \frac{1}{3} - \frac{1}{300} = \frac{1}{3} - \frac{1}{3\cdot(10)^2}, \\ S_3 &= \frac{333}{1000} = \frac{1}{3} - \frac{1}{3000} = \frac{1}{3} - \frac{1}{3\cdot(10)^3}, \\ S_4 &= \frac{3333}{10{,}000} = \frac{1}{3} - \frac{1}{30{,}000} = \frac{1}{3} - \frac{1}{3\cdot(10)^4}, \end{aligned}$$

and, in general,

$$S_n = \frac{1}{3} - \frac{1}{3\cdot(10)^n}.$$

If we take n larger and larger, which means that we take more and more terms of $0.3333 \cdots$, we get a terminating decimal fraction closer and closer to $\frac{1}{3}$. If we stop at any term, such as 0.3333333333, this terminating decimal is *not* equal to $\frac{1}{3}$. However, it differs from $\frac{1}{3}$ by only $\frac{1}{3\cdot(10)^{10}}$. We say that this terminating decimal represents $\frac{1}{3}$ correct to 10 decimal places.

For the fraction $2/7$ we obtain the non-terminating decimal $0.285714\,285714\,285714 \cdots$. For both $1/3$ and $2/7$ we obtain a *periodic* or *repeating* decimal, that is, a decimal in which a group of digits keeps repeating over and over. It is not necessary that the repeating group start immediately after the decimal point; for instance, $3/700 = 0.00428571\,428571 \cdots$ and $37/30 = 1.23333 \cdots$.

We may, if we wish, consider a terminating decimal such as 0.575 a repeating decimal since, as we have noted, this may be written as $0.575000 \cdots$. Here the group of repeating digits consists of the digit 0 only.

It can be shown that every rational number may be written as a repeating decimal (including "terminating decimals") and that every repeating decimal is a rational number. This, of course, raises the question: What about non-repeating decimals? What, if anything, does a symbol such as $0.20220222022220 \cdots$ mean, in which each group of 2's has one more digit in it than the preceding group. Or what, if anything, does it mean when we write the symbol $3.14159\,26535\,89793\,23846 \cdots$ in which there does not appear to be even as much "law" for the digits as in the preceding example? Whatever these last two symbols may mean, they do not represent rational numbers.

We have seen how to change a rational number written in the form a/b to the form of a (repeating) decimal. The reverse process, changing a (repeating) decimal to the form a/b, is easily accomplished.* Consider the repeating decimal $0.123123123 \cdots$. We let

$$S = 0.123123123 \cdots.$$

Then

$$1000 \cdot S = 10^3 \cdot S = 123.123123123 \cdots$$

and

$$10^3 \cdot S - S = (123.123123123 \cdots) - (0.123123123 \cdots),$$
$$S(10^3 - 1) = 123,$$
$$S = \frac{123}{10^3 - 1} = \frac{123}{999} = \frac{41}{333}.$$

(We may indicate the group of repeating digits more briefly by placing a vinculum over them; thus $0.123123123 \cdots = 0.\overline{123}$.)

---

* A justification of the steps we take cannot be given here. Complete justification would require some knowledge of what are called "infinite series."

In general, if we have a repeating decimal which we wish to write in the form a/b and there are n digits in the repeating group, then, if S is the value of the repeating decimal, we find $10^n \cdot S$. This will be the same as the given number except that the decimal point has been moved n places to the right. We find $10^n \cdot S - S$, and from it we can find S as a ratio of two integers.

Exercise 17.3.1

Write as the ratio of two integers:

1. $0.\overline{6}$.
2. $5.\overline{6}$.
3. $0.\overline{10}$.
4. $3.2\overline{45}$.
5. $0.00\overline{23}$.
6. $0.\overline{857142}$.
7. $8.\overline{571428}$.
8. $76.\overline{923076}$.

17.4 Scientific Notation

"· · · the weight of the lightest atom, hydrogen, is found to be $(1.67)10^{-24}$ gram, while that of the heaviest, naturally occurring element, uranium, is $(3.95)10^{-22}$ gram. Special balances, designed for weighing small quantities, can detect a weight as small as 10^{-8} gram. Such a minute speck of matter, say of uranium, would be invisible to the naked eye, but it would, nevertheless, contain more than 10^{13} atoms." *

If we should use our decimal notation to write the numbers given in this quotation, they would appear as follows:

$$(1.67)10^{-24} = 0.000\,000\,000\,000\,000\,000\,000\,00167,$$

$$(3.95)10^{-22} = 0.000\,000\,000\,000\,000\,000\,000\,395,$$

$$10^{-8} = 0.000\,000\,01,$$

$$10^{13} = 10{,}000{,}000{,}000{,}000.$$

Not only do such numbers take up a large amount of page space when they are written in the decimal notation but they certainly are difficult to read (you might try reading the first of these numbers). When in the decimal form, they are also difficult to use in calculations.

---

* Glasstone: *Sourcebook on Atomic Energy,* copyright 1950, Atomic Energy Commission, D. Van Nostrand Company, Inc.

For these reasons scientists usually do not use the decimal notation when working with such small or such large numbers. They use the so-called scientific or standard notation. This notation depends upon the following fact:

Any number can be written as a number not less than 1 *but less than* 10, *multiplied by an integral power of* 10.

EXAMPLES 17.4.1

1. The examples from our quotation.
2. $528{,}000{,}000 = (5.28)10^8$.
3. The wavelength of red light is about 0.000076 centimeter $= (7.6)10^{-5}$ centimeter.
4. The wavelength of violet light is about $(3.85)10^{-5}$ centimeter.
5. A light-year (the distance light will travel in one year at the rate of $186{,}000 = (1.86)10^5$ miles per second) is about $5{,}880{,}000{,}000{,}000 = (5.88)10^{12}$ miles. (If given in inches it might be more impressive.)
6. $\dfrac{(2.3)10^{-8}\cdot(3.2)10^6}{(6.4)10^{-12}} = \dfrac{(2.3)(3.2)\cdot 10^{10}}{6.4} = \dfrac{2.3}{2}\cdot 10^{10}$ $= (1.15)10^{10}$.

17.4.2 DEFINITION: The exponent of the power of 10 which appears when a number is written in the scientific (standard) notation is called the *characteristic* of the number.

To determine the characteristic of a number which has been written in decimal notation and which we wish to write in the scientific notation we follow the rule:

17.4.3 Count the number of places which the decimal point must be moved in order to bring it to the position it has in the scientific notation; this gives the absolute value (size) of the characteristic. Then, if the point must be moved to the *left* to attain this position, the characteristic is *positive;* if the point must be moved to the *right* to attain this position, the characteristic is *negative.*

EXERCISE 17.4.4

1. Verify that the numbers above which are written in both the decimal and scientific notations have the correct characteristics.

Write in the scientific notation:

2. 5,000,000.
3. 32,000,000.
4. 573,000,000,000.
5. 60,100,000,000,000.
6. 0.002.
7. 0.000035.
8. 0.000 000 000 00123.
9. 6.74.

Write in decimal notation:

10. $3 \cdot 10^7$.
11. $(3.6)10^9$.
12. $(5.24)10^6$.
13. $(4.1)10^{-5}$.
14. $(5.03)10^{-9}$.
15. $(6.214)10^{-13}$.

Calculate and express the result in both scientific and decimal notations:

16. $\dfrac{(6 \cdot 10^7)(4 \cdot 10^{-4})}{8 \cdot 10^{-5}}$.

17. $\dfrac{(5.1)10^{-5} \cdot (2.4)10^{17}}{(3.4)10^{12} \cdot (6.0)10^6}$.

18. About how many atoms are there in 10^{-8} gram of hydrogen?

19. One gram is about 15.4 grains and there are 7000 grains to the pound. About how many pounds does an atom of uranium weigh? About how many atoms are there in a pound of uranium?

18

Equations and Their Solutions

We have already solved some simple types of equations. Each time that we solved an equation we did it by making use of the definitions of the inverse operations of subtraction and division. We shall now investigate more systematically the general problem of equation solving.

18.1 Identical and Conditional Equations

We stated earlier that an equation is a mathematical sentence. In English a sentence may be either a statement or a question. Statements may be classified as particular or general. A particular statement concerns one member of a set of things; a general statement concerns each member of a set of things. Furthermore, statements may be true or they may be false.

EXAMPLES 18.1.1

1. Abraham Lincoln was the sixteenth president of the United States. This is a particular statement, and it is true.

2. John Tyler was the sixteenth president of the United States. This is a particular statement, and it is false.

3. Presidents of the United States were more than 35 years old when elected president. This is a general statement. It remains true if the subject "Presidents of the United States" is replaced by any name from the set of names of presidents. It becomes meaningless if we replace the subject with "Alexander Hamilton." Alexander Hamilton is not in the set of names which make the statement have meaning.

4. $4 + 1 = 5$. This is a particular statement, and it is true.

5. $7 + 1 = 5$. This is a particular statement and it is false.

6. $\dfrac{1}{x-1} + 1 = \dfrac{x}{x-1}$. This is a general statement. It remains a true statement if x is replaced by any number in the set of numbers which make the expressions on either side of the equals sign meaningful. It becomes meaningless if x is replaced by 1. The number 1 is not in the set of numbers which make the statement have meaning. Or, if we look upon each side of the equation as a function of x, we may say that 1 is not in the range of the variable x for these functions.

7. Who was the sixteenth president of the United States? This is a question. (The question mark is really not essential to our realization that this is a question.) If we replace the "who" with "Abraham Lincoln," it becomes a true statement; if we replace "who" with "John Tyler," it becomes a false statement.

8. $x + 1 = 5$. This is a question. It asks, "Is there a number x which makes this a true statement and, if so, what is it?" If we replace the x with 4, it becomes a true statement; if we replace the x with 7, it becomes a false statement.

9. Who was the president who was more than 8 feet tall? This is a question. If we replace "who" by any name from the list of names of presidents, it becomes a false statement.

10. $1 + 1/x = 1$. This is a question: Is there a number x which makes this a true statement, and, if so, what is it? We shall discover that there is no number x which makes this a true statement.

If we reconsider the equations which we obtained for the verbal problems that we have had, such as Problem 4 of 3.1.2, we see that in setting up these equations we equate two functions. In the problem referred to, we have the function which expresses "four times three more than a certain number, say x" equated to the function which expresses "36 more than the (same) number." That is, we have $4(x + 3) = x + 36$.

18.1.2 **Definition**: An equation is an indicated equality between two functions.

18.1.3 **Definition**: If an equation becomes a true particular statement for each value of the independent variable for which both functions have meaning, the equation is said to be an *identity* or an *identical equation*.

18.1.4 DEFINITION: If an equation becomes a false particular statement for at least one value of the independent variable, the equation is said to be a *conditional equation.*

We must be very careful to distinguish between a false statement and a meaningless statement. It would be well to read the above Examples 2 and 5 and 3 and 6 again.

EXAMPLES 18.1.5

1. $(x + 1)^2 = x^2 + 2x + 1$ is an identity. It becomes a true particular statement for every value of x. We can show this by starting with the statement, "Let x be any number," and then showing that the function on the left side of the equals sign can be written in the form of the function on the right side or by showing that the right side can be written in the form of the left side.

2. $x + 1 = 5$ is a conditional equation. It becomes a false statement if x is replaced with 7. There are many other values of x which make this a false statement. There is only one value which makes it a true statement. However, this last is irrelevant to the fact that it is a conditional equation. If there were no values of x which make it a true statement, it would still be a conditional equation.

3. $(x - 1)(x - 2)(x - 3)(x - 4)(x - 5) = 0$ is a conditional equation. It becomes a true statement if x is replaced with 1, 2, 3, 4, or 5; it becomes a false statement if x is replaced with 6.

4. $\dfrac{x}{x^2 - 4} + 1 = \dfrac{x^2 + x - 4}{x^2 - 4}$ is an identity. For what values of x does it become a meaningless statement?

EXERCISE 18.1.6

Determine which of the following are identities and which are conditional equations. For which values of the variable do the identities become meaningless?

1. $(x - 3)(x + 3) = x^2 - 9.$
2. $(x - 3)(x + 3) = 0.$
3. $x^2 - 4x + 4 = (x - 2)^2.$
4. $(x - 3)^2 = x^2 + 6x + 9.$
5. $t^2 - 16 = (t - 4)^2.$

6. $6x^2 + 13x - 5 = (3x - 1)(2x + 5)$.

7. $s^2 + 11s + 10 = (s + 2)(s + 5)$.

8. $6p^2 + 7p + 2 = (3p + 2)(2p + 1)$.

9. $\dfrac{1}{x - 2} - \dfrac{1}{x - 3} = \dfrac{-1}{x^2 - 5x + 6}$.

10. $\dfrac{1}{r - 1} - \dfrac{1}{r + 1} = \dfrac{1}{r^2 - 1}$.

11. $\dfrac{1}{s - 1} \cdot \dfrac{1}{s + 1} = \dfrac{1}{s^2 - 1}$.

12. $(x - 1)(x + 1)(x^2 + 1) = x^4 - 1$.

13. $(x - 2)^2(x + 2)^2 = x^4 - 8x^2 + 16$.

14. $\dfrac{1}{(x + 2)^2} + \dfrac{1}{x^2 - 4} = \dfrac{2x}{(x - 2)(x + 2)^2}$.

15. Make up a conditional equation which becomes a true statement for ten different values of the variable.

16. Make up a conditional equation which becomes a true statement for 1000 different values of the variable.

18.1.7 Definition: Those values of the variable (or unknown) which make the equation a true statement are called *solutions* or *roots* of the equation.

18.2 Equivalent Equations

Examples 18.2.1

1. Who was the sixteenth president of the United States?
   Who was the president who immediately preceded Andrew Johnson?
   Who was the president who immediately followed James Buchanan?
   Who was the president who was assassinated by John Wilkes Booth?

If in each of these questions we replace the "who" with "Abraham Lincoln," we obtain a true statement. Because each of these questions requires the same "value" for "who" in order to make it a true statement, we say these questions are *equivalent*.

2. $x + 1 = 5,$

$$2x + 3 = x + 7,$$

$$3(x - 2) = 2(x + 1) - \frac{4(x - 1)}{3},$$

$$x = 4,$$

$$\frac{1}{x + 1} + \frac{1}{x - 1} = \frac{8}{5(x - 1)}.$$

We note that 4 is the only solution (or root) of each of these equations.

3. $(x - 1)(x + 1) = 0,$

$$x^2 - 4 = 5 - (x^2 + 7),$$

$$\frac{x^3 + 4x - 2}{x^2 - 2x} = \frac{4}{x - 2} + \frac{1}{x}.$$

We note that $+1$ and -1 are the only roots of each of these equations. We also note that the question, Is there a number x such that either $x = 1$ or $x = -1$? is equivalent to each of these conditional equations.

18.2.2 DEFINITION: Two (or more) equations which have the same solution (or solutions) are said to be *equivalent equations.*

This means that two equations are equivalent if, and only if, every solution of the first equation is a solution of the second equation and every solution of the second equation is a solution of the first equation. The equations $x - 1 = 0$ and $x^2 - 1 = 0$ are *not* equivalent. The only solution of the first is 1, whereas the second has the solutions 1 and -1. Every solution of the first is a solution of the second, but not every solution of the second is a solution of the first.

Equivalent equations play a major role in the process of finding the roots of an equation. There are several ways in which we may attempt to find the roots. One way would be to guess. If we make a good guess, indicated by the fact that our guess makes a true statement of the equation, we may cheer. Also we might try crystal gazing. Neither of these methods is very satisfactory in most cases. The systematic method is to obtain, if we can, a sequence of equations such that every solution of the given equation must be a solution of the others and such that each equation is simpler in form than its predecessor, until we finally have an equation whose roots we can find by inspection. Usually these final equations are of the form $x =$ (a constant).

If we know not only that every solution of the first equation is a solution of the last but also that every solution of the last is a solution of the first (that is, if we know that the first and last equations are equivalent), then we know the solutions of the given equation as soon as we know the solutions of the last equation. In certain cases there may be solutions of the last equation which are not solutions of the given equation. But by taking certain precautions we can always be sure that all the solutions of the given equation are *among* the solutions of the last equation.

To obtain this sequence of equations we make use of the following rules:

18.2.3 If a, b, c, d are numbers such that $a = b$ and $c = d$, then

(1) $a + c = b + d$. (If equals are added to equals, the sums are equal.)
(2) $a - c = b - d$. (If equals are subtracted from equals, the differences are equal.)
(3) $a \cdot c = b \cdot d$. (If equals are multiplied by equals, the products are equal.)
(4) $a/c = b/d$, if c and d are not zero. (If equals are divided by equals, the quotients are equal.)

EXAMPLES 18.2.4

1. To find the solutions of the equation $7x - 2 = 3x + 6$: Suppose that there is a number x such that

$$7x - 2 = 3x + 6. \tag{a}$$

Then for this number x

$$7x = 3x + 8. \tag{b}$$

(We added 2 to each side.) Then for this number x

$$4x = 8. \tag{c}$$

(We subtracted $3x$ from each side.) Then for this number x

(*d*) $$x = 2.$$

(We divided each side by 4.)

We have now shown that, if there is a number x which makes (*a*) a true statement, then this number must be 2. No other number can possibly make (*a*) a true statement, and, as far as we know at this point, even 2 may not make it a true statement. But by replacing x with 2 in (*a*) we see that 2 actually is a root, and since it is the only possible root it must be the only root. Also we may now state that each of the equations (*a*), (*b*), (*c*), and (*d*) is equivalent to each of the others.

2. Find the solutions of $x - 2 + \dfrac{1}{x-1} = 1 - \dfrac{x-2}{x-1}$. Suppose that there is a number x such that

(*a*) $$x - 2 + \frac{1}{x-1} = 1 - \frac{x-2}{x-1}.$$

Then for this number x

(*b*) $$(x-2)(x-1) + 1 = (x-1) - (x-2).$$

(We multiplied each side by $x - 1$.) Then for this number x

(*c*) $$x^2 - 3x + 3 = 1.$$

(Here we have merely multiplied out and collected like terms on each side.) Then for this number x

(*d*) $$x^2 - 3x + 2 = 0.$$

(We subtracted 1 from each side.) Then for this number x

(*e*) $$(x-1)(x-2) = 0.$$

(Here we have merely factored the polynomial on the left side.) Then for this number x

(*f*) $$\text{Either } x - 1 = 0 \text{ or } x - 2 = 0.$$

We see that the only numbers which can possibly make (*a*) a true statement are 1 or 2 or both.

If we replace x with 1 in equation (*a*), we find that 1 is not a solution, but, if we replace x with 2, we find that 2 is a solution.

Equation (*e*) is not equivalent to (*a*), but every solution of (*a*) is among the solutions of (*e*). Which of the equations in this last example are equivalent?[29]

3. Suppose that there is a number x such that

(*a*) $$x^2 = x + 2.$$

Then for this number x

(*b*) $$x^2 - 1 = x + 1.$$

(What did we do here?)[30] Then for this number x

(*c*) $$x - 1 = 1.$$

(We divided each side by $x + 1$.) Then for this number x

(*d*) $$x = 2.$$

Now 2 is a solution of (*a*). But, if we should go about the problem in a different manner, we should find that -1 also is a solution. It is not true that all the solutions of (*a*) are among the solutions of (*d*). The trouble arose in the step that we took in going from (*b*) to (*c*). We divided by $x + 1$, and now we see that this may be zero, i.e., if $x = -1$.

4. Suppose that there is a number x such that

(*a*) $$x + 1 + \frac{1}{x - 4} = 5 + \frac{1}{x - 4}.$$

Then for this number x

(*b*) $$x + 1 = 5.$$

(Subtracting $\frac{1}{x - 4}$ from each side.) Then for this number x

(*c*) $$x = 4.$$

But 4 is not a solution of the first equation. Why?[31] Equations (*a*) and (*c*) are not equivalent.

These examples point out the precautions we must observe in obtaining our sequence of equations:

1. We may be certain that the roots of the original equation are among the roots of the final equation, if we have at no step

divided each side by an expression which becomes zero for some value of the variable. (Ex. 3.) *

2. We may multiply each side by an expression which becomes zero for some value of the unknown. But then the final equation may have roots which are not roots of the original equation. (Ex. 2.) In this case we must *always* test each root of the final equation in the original equation.

3. We may be certain that the original and final equations are equivalent: (*a*) If at each step we do not add or subtract an expression which may become meaningless for some value of the variable. (Ex. 4.) (Adding or subtracting a constant or a polynomial will always produce an equivalent equation, since a polynomial cannot become meaningless for any value of the variable.) (*b*) If at each step we do not multiply or divide by anything but a constant (not zero). (Ex. 1.)

We may always perform any indicated operations on either side of an equation. This does not change either function except for the forms in which they are written.

EXAMPLES 18.2.5

1. Solve the equation $x/2 + 3 = 7$. Suppose there is a number x such that

$$\frac{x}{2} + 3 = 7. \tag{a}$$

Then for this number x

$$\frac{x}{2} = 4. \tag{b}$$

(We subtracted 3 from each side.) Then for this number x

$$x = 8. \tag{c}$$

(We multiplied each side by 2.)

Now we may be sure that all these equations are equivalent since we have subtracted and multiplied by constants only. Hence (if we have not made a mistake) 8 must be a root and the only root of the original equation, since it is a root and the only root of the final equation. To be sure that we have not made a mistake we might check in the original equation.

---

* In practice it is very seldom a good idea to divide each side of an equation by an expression which involves the variable.

2. Solve the equation $1 - \frac{1}{x-2} = \frac{1}{(x-2)(x-3)}$. Suppose there is a number x such that

$$1 - \frac{1}{x-2} = \frac{1}{(x-2)(x-3)}. \tag{a}$$

Then for this number x

$$(x-2)(x-3) - (x-3) = 1. \tag{b}$$

(We multiplied each side by $(x - 2)(x - 3)$.) This must act as a warning. Since we have multiplied each side by an expression involving the variable, the final equation may have roots which are not roots of the original equation.) Then for this number x

$$x^2 - 6x + 9 = 1. \tag{c}$$

(We simply multiplied out and combined like terms on each side.) Then for this number x

$$x^2 - 6x + 8 = 0. \tag{d}$$

(We subtracted 1 from each side.) Then for this number x

$$(x-4)(x-2) = 0. \tag{e}$$

(We factored the left side.) Then for this number x

$$x - 4 = 0 \text{ or } x - 2 = 0. \tag{f}$$

Then for this number x

$$x = 4 \text{ or } x = 2. \tag{g}$$

Since at one step we multiplied each side by an expression involving the variable, it is not necessarily true that 4 and 2 or either are roots of the given equation. The best that we can say at this point is that, *if* the given equation has roots, they are either 4 or 2 or both.

By trial we find that 4 is a root but that 2 is not. We may now state that 4 is the only root of the given equation. Which of the equations (*a*), (*b*), (*c*), (*d*), (*e*), (*f*), (*g*) are equivalent?[32]

3. Solve the equation $2x = \dfrac{x^2 - 7x + 4}{x - 5} + 2$. Suppose there is a number x such that

(*a*) $$2x = \frac{x^2 - 7x + 4}{x - 5} + 2.$$

Then for this number x

(*b*) $$2x^2 - 10x = x^2 - 7x + 4 + 2x - 10.$$

(We multiplied each side by $x - 5$. Remember that this is a warning!) Then for this number x

(*c*) $$2x^2 - 10x = x^2 - 5x - 6.$$

(We combined like terms on the right side.) Then for this number x

(*d*) $$x^2 - 5x + 6 = 0.$$

(We subtracted $x^2 - 5x - 6$ from each side.)

(*e*) $$(x - 2)(x - 3) = 0.$$

(*f*) $$x = 2 \text{ or } x = 3.$$

By trial we find that both 2 and 3 are roots. Which of these equations are equivalent?[33]

EXERCISE 18.2.6

Solve the following equations. State what you did at each step. Tell which of your equations are equivalent.

1. $2x = 10$.

2. $3x = -12$.

3. $-4x = 8$.

4. $3t = 7$.

5. $2s - 3 = 5$.

6. $7s + 11 = 3s + 27$.

7. $3w - 2 = 5w + 7$.

8. $3(x + 1) + 2 = 2(x - 1) + 5$.

9. $4[x + 3(x - 5)] = 2 + 3[1 - 2(x + 1)]$.

10. $3(x + 2) + x - 3 = 8(x + 1) - 2[2(x + 3) - 3]$.

11. $6[x + 3(x - 2)] - 7 = 2[x - 5(1 - x)] + 3(2x - 11)$.

12. $\dfrac{x}{2} - 6 = x - 3$.

13. $\dfrac{s}{3} - 2 = s + \dfrac{1}{6}$.

14. $0.2x + 2.4 = 5.2.$

15. $3.4p + 5.1 = 2.3p + 11.7.$

16. $\dfrac{3x + 2}{2} = x + 5.$

17. $\dfrac{5k - 1}{2} = \dfrac{2k + 1}{4}.$

18. $\dfrac{2t + 1}{7} = t - 1 + \dfrac{t + 1}{14}.$

19. $9(x + 3) = 36(x - 3).$

20. $7x + 14 + 21(x - 2) = 0.$

21. $3(x - 2) = x(x - 2).$

22. $\dfrac{2}{x - 1} = 1.$

23. $\dfrac{2}{x - 1} = 1 + \dfrac{1}{x - 1}.$

24. $\dfrac{2}{x - 1} = 1 + \dfrac{2}{x - 1}.$

25. $\dfrac{2}{x - 2} - 2 = \dfrac{x}{x - 2}.$

26. $x + \dfrac{2x}{x - 3} = \dfrac{6}{x - 3}.$

27. $2x + 2 - \dfrac{3}{x} = x + \dfrac{5}{x}.$

28. $2x = \dfrac{x}{x - 2} + x.$

18.3 Paths to Solutions

A question which always arises when you set out to solve a mathematical or other kind of problem is, "What steps should I take to arrive at a solution of this problem?" The answer to this is not always unique. There may be several paths that you may follow, some shorter than others.

Certainly, before you attempt to answer the question *how* to get to your goal, a solution of your problem, you must first decide *where* you want to go. You must not be like Alice in *Alice in Wonderland.* Alice asked the Cheshire Cat:

> "Would you tell me, please, which way I ought to go from here?"
> "That depends a good deal on where you want to get to," said the Cat.
> "I don't much care where—," said Alice.
> "Then it doesn't matter which way you go," said the Cat.

In solving an equation your goal is to obtain an equation or several equations of the form $x =$ (a constant) such that all the solutions of the given equation are among the solutions of the final equation or equations.

After you have decided where you want to go, you may consider several possible paths and decide upon one of them. As you take each step down the path you should always have in mind two questions:

1. Is this step mathematically correct? Is it in accord with all the rules that we have agreed upon for our (mathematical) game?
2. Does this step get you nearer your goal?

The answer to the first question *must* always be "yes."

The answer to the second question, if you want to solve the problem as efficiently as possible, should be "yes."

Examples 18.3.1

1. Solve the equation $2x + 3 = x + 5$. Suppose that there is a number x such that

$$2x + 3 = x + 5. \tag{a}$$

Then for this number x

$$2x + 7 = x + 9. \tag{b}$$

(We added 4 to each side.) Then for this number x

$$5x + 7 = 4x + 9. \tag{c}$$

(We added $3x$ to each side.) Then for this number x

$$5x - 2 = 4x. \tag{d}$$

(We subtracted 9 from each side.) Then for this number x

$$5x = 4x + 2. \tag{e}$$

(We added 2 to each side.) Then for this number x

$$x = 2. \tag{f}$$

(We subtracted $4x$ from each side.)

We have finally arrived at our goal, and we may say (in this case) that 2 is a solution and the only solution of our given equation. Each step was mathematically correct. But we have surely not taken the shortest path! Actually this equation can be solved in one step.

2. Solve the equation $\dfrac{8x + 3}{2x + 1} = 3$. Suppose that there is a number x such that

$$\frac{8x + 3}{2x + 1} = 3. \tag{a}$$

Then for this number x

$$\frac{4x + 3}{x + 1} = 3. \tag{b}$$

(We divided out the common factor 2 from $8x$ and $2x$.) Then for this number x

$$4x + 3 = 3x + 3. \tag{c}$$

(We multiplied each side by $x + 1$.) Then for this number x

$$x = 0. \tag{d}$$

(We subtracted $3x + 3$ from each side.)

By trial we find that 0 is actually a solution of our given equation. But our work is worthless! It has been our misfortune to obtain a correct result by an incorrect method. (At which step did we use an incorrect operation?) [34]

If in solving a problem we make an error in calculation and obtain an incorrect result after using correct methods, that is mildly unfortunate. But to obtain a correct result by incorrect methods is tragic. Since our method "worked" in this case, we may be convinced that it is a correct method. To become unconvinced (if we do not realize our error for other reasons) we might try the "method" on the equation $\dfrac{8x + 3}{2x + 1} = 2$.

You must be the one to determine which path you will follow. The best we can do is to give you some general advice.

1. If there are some indicated operations on either or both sides of the equation, carry them out and combine like terms.

2. If the functions on each side of the equation are polynomials, subtract one of the polynomials from each side of the equation so that you obtain an equation of the form $P(x) = 0$. Then, if you can, factor the polynomial into first-degree factors and make use of 7.1.6.

3. If $ax + b = 0$, by subtracting b from each side and then dividing each side by a we find $x = -b/a$.

4. If the equation contains fractions, multiply each side by the LCD of all the fractions. (Remember that the resulting equation may not be equivalent to the original equation if the LCD involves the variable.)

Exercise 18.3.2

Solve the following equations:

1. $5x + 3 = 2x + 9.$
2. $(2x - 7) - (x - 5) = 0.$
3. $3(2x - 7) = 5 - (1 - x).$
4. $9(4 - 5x) + 2 = 7(6x - 7).$
5. $5 - (6 + 9x) = 9 - (4x - 1).$
6. $10 - 4(3x - 1) = 6(x + 11) + 2x.$
7. $7 = 6y + 2(-3 + 4y) + 4(3 - y) + 11.$
8. $3(t + 2) = 2(-t - 6) + 3t - (-t - 4).$
9. $2(2x + 3) - 3(1 - x) - 2(2x - 5) = 5(2x - 1) + 7(5 - x).$
10. $3[1 + 2(1 - x)] - 4[3 - (x + 3)] = 5\{x + 2[1 - 2x - 3(1 - 2x)] + 4\}.$
11. $(x - 3)(x - 2) = (x + 3)(x - 2).$
12. $(2x + 1)(x - 3) = (2x + 3)(x - 1).$
13. $(4x - 3)(2x + 5) = (8x - 1)(x + 2).$
14. $(3x + 5)(4x + 3) = (6x - 5)(2x + 7).$
15. $x - 2 - \dfrac{3x - 2}{2} = \dfrac{1}{2} + x.$
16. $4 - 3t + \dfrac{2t - 3}{5} = \dfrac{4t + 5}{2}.$
17. $\dfrac{2(2s - 5)}{3} - \dfrac{6s + 5}{6} = \dfrac{4s - 7}{6}.$
18. $\dfrac{5[r - 3(r - 1)]}{2} - \dfrac{1}{3} = \dfrac{7(r - 4)}{12} + 1.$
19. $\dfrac{1}{x - 1} + \dfrac{2}{x + 1} = \dfrac{4}{x - 1}.$
20. $\dfrac{1}{x - 1} + \dfrac{2}{x + 1} = \dfrac{3}{x - 1}.$
21. $\dfrac{3}{x - 2} + \dfrac{1}{x + 2} = \dfrac{8}{x^2 - 4}.$
22. $\dfrac{3}{x - 2} + \dfrac{1}{x + 2} = \dfrac{12}{x^2 - 4}.$

23. $\dfrac{4}{x+2} - \dfrac{3}{x+3} = \dfrac{8}{x^2+5x+6}.$

24. $\dfrac{4}{x+2} - \dfrac{3}{x+3} = \dfrac{3}{x^2+5x+6}.$

25. $\dfrac{2x+8}{x+5} = 1.$

26. $\dfrac{8x+3}{2x+1} = 2.$

27. $\dfrac{3x-2}{2x+4} = \dfrac{1}{2}.$

28. $\dfrac{4x+1}{1-2x} = \dfrac{3}{11}.$

29. $\dfrac{x+1}{x-1} = \dfrac{2x+1}{2x-1}.$

30. $\dfrac{2(x+1)}{x-3} = \dfrac{8}{x-3}.$

31. $\dfrac{3}{x} - \dfrac{x-2}{x-3} = \dfrac{x-4}{3-x}.$

32. $\dfrac{4}{x} - \dfrac{3x+2}{x+1} = \dfrac{2-3x}{x+1}.$

33. $\dfrac{3}{x} - \dfrac{3x-2}{x+1} = \dfrac{2-3x}{x+1}.$

34. $\dfrac{2}{x} - \dfrac{3x+2}{x+1} = \dfrac{2-3x}{x+1}.$

35. $\dfrac{3x-2}{x-1} = 5 + \dfrac{1}{x-1}.$

36. $\dfrac{3x-2}{x-1} = 5 + \dfrac{2}{x-1}.$

37. $\dfrac{3x-2}{x-1} = 3 + \dfrac{2}{x-1}.$

38. $\dfrac{x-1}{x+1} + x = 3 - \dfrac{2}{x+1}.$

39. $\dfrac{x-1}{x+1} + x = 2 - \dfrac{2}{x+1}.$

40. $\dfrac{x-1}{x+1} + x = 1 - \dfrac{2}{x+1}.$

41. $\dfrac{5}{x} - \dfrac{3}{x-1} = \dfrac{2}{x+1}.$

42. $\dfrac{5}{x-1} - \dfrac{7}{x-2} + \dfrac{2}{x-3} = 0.$

43. $\dfrac{2}{x^2-1} = \dfrac{3}{x^2+x-2}.$

44. $\dfrac{2}{x^2-1} = \dfrac{4}{x^2+x-2}.$

45. $\dfrac{4}{6x^2-11x+3} = \dfrac{5}{8x^2-18x+9}.$

46. $\dfrac{2}{2x^2+x+2} = \dfrac{3}{3x^2+x+5}.$

47. $\dfrac{11}{x^2+6} = \dfrac{6}{x^2-4}$.

48. $x - 2 = \dfrac{x-6}{x+4}$.

49. $x - 5 + \dfrac{5x}{x+2} = \dfrac{x-8}{x+2}$.

50. $\dfrac{x+2}{2} = \dfrac{x+6}{x+2}$.

51. $3 - x - \dfrac{9}{6x+5} = \dfrac{8x}{6x+5}$.

52. $\dfrac{1}{x^2-1} = \dfrac{2}{x^2-x}$.

53. $\dfrac{x-2}{x+6} - \dfrac{x+2}{x-4} = \dfrac{4x-76}{x^2+2x-24}$.

54. $x^2 + \dfrac{1}{x+1} = \dfrac{2x+1}{x+1}$.

55. $\dfrac{1+x^3}{(1+x)^3} = \dfrac{1}{3}$.

56. $\dfrac{x^3-8}{(x-2)^3} = \dfrac{1}{3}$.

57. $4[4x(x+1) + 2(2+x) - 3] = 24(x+1) + 5$.
58. $5x(x+1) = [5(x+1) - 6x][6x - 5(x-1)]$.
59. $x^3 - 3x^2 = 28x$.
60. $3x[3(x-1) - 2(x+1)] + 4x = 4(x-1) - 3x(2x+1)$.

61. $(3x^2-2)^2 - x^2 = 0$.
62. $(5x^2-3)^2 = 4x^2$.
63. $x^4 = 5x^2 - 4$.
64. $x^4 = 13x^2 - 36$.

Set up the equations and solve them:

65. Twenty years ago when John and Mary were married John was twice as old as Mary. Now the sum of their ages is 115 years. What are their present ages?

66. A board 12 feet long is cut into two pieces, one piece being 6 inches shorter than four times the length of the other piece. How long is each piece?

67. Albert has \$12.95 in coins. He has 2 less than three times as many quarters as half dollars and twice as many dimes as quarters. How many does he have of each?

68. Paul has \$4.50 in coins. He has 3 less than twice as many quarters as half dollars and three times as many dimes as quarters. How many does he have of each?

69. A grocer wanted to blend some coffee selling at 80 cents a pound with some selling at 95 cents a pound so as to obtain 75 pounds to sell at 90 cents a pound. How much of each should he use?

70. A grocer wanted to blend some coffee selling at 87 cents a pound with some selling at 95 cents a pound so as to obtain 75 pounds to sell at 90 cents a pound. How much of each should he use?

71. A passenger train and a freight train left towns 320 miles apart and traveled towards each other. The average speed of the passenger train was 20 mph faster than the average speed of the freight train. If they met after 4 hours what were their average speeds?

72. A car leaves town at noon and travels at 40 mph. At 1 o'clock another car leaves on the same route and travels at 50 mph. When will the second car be 10 miles behind the first? When will it pass the first? When will it be 10 miles ahead of the first?

73. A man divided his property worth \$5300 among his four sons, Albert, Byron, Carl, and David. He gave Byron \$350 more than he gave Albert; he gave Carl \$400 more than he gave Byron; and he gave David twice as much as he gave Albert and Byron together. How much did each receive?

74. A man divided his property worth \$1872 among his wife, three sons, and two daughters. His wife received three times as much as either daughter, and each son received half as much as a daughter. How much did each receive?

75. A picture measures 18 inches by 12 inches. A frame of uniform width is to be constructed for it so that the area of the frame equals the area of the picture. How wide should the frame be?

76. The difference between two numbers is 9, and the larger number times their sum is 266. What are the numbers?

77. Two numbers differ by 12, and their sum times the larger is 560. What are the numbers?

78. At noon an airplane flying due north at 200 mph passes over an airport. At 1 o'clock another plane passes over the air-

port flying east at 300 mph. When will the distance between the planes be 500 miles?

79. A snail is moving along the y-axis at the rate of 4 inches per hour and passes the origin at 9 A.M. Another snail is moving along the x-axis at 5 inches per hour, and it passes the origin at 11 A.M. When will they be 13 inches apart?

80. The minute hand and the hour hand of a clock are together at noon. What is the next time that they are together?

81. A square piece of tin has a 3-inch square cut from each corner. The sides are then turned up to form a pan 3 inches deep. What was the length of the sides of the original piece of tin, if the pan will hold 108 cubic inches?

82. The difference between a number and its reciprocal is $^{16}/_{15}$.

83. If $^{77}/_{18}$ is added to a certain number, we obtain the reciprocal of the number.

84. A hardware dealer sold a bicycle for $39. His per cent profit equaled the cost to him of the bicycle in dollars. What was the cost of the bicycle?

85. A stone is projected upward from the roof of a building 200 feet high at an initial velocity of 40 feet per second. After t seconds it will be $40t - 16t^2$ feet above the roof. When will it be 24 feet above the roof? When will it again be level with the roof? When will it strike the ground?

86. A group of people had dinner together in a restaurant. The total bill came to $17.50. As the waiter was bringing the check two of the people suddenly discovered they had urgent business elsewhere. Those who were stuck had to pay $1 more each than their fair shares. How many people were in the original group?

87. To pave a room with square tiles of a certain size, 360 tiles were needed. If the tiles had been 1 inch larger on each side, only 250 of them would have been needed. What was the size of the tiles?

88. The sum of the digits of a 2-digit number is 13. If the digits are reversed, the resulting number is 45 more than the original number.

89. The units digit of a 2-digit number is 5 more than the tens digit. The number itself is three times as great as the sum of its digits.

90. In a 2-digit number the tens digit is 1 less than the units digit. If the digits are interchanged, the sum of the new number and the original number is 160.

91. The tens digit of a 2-digit number is 2 more than the units digit and the number itself is 1 more than the sum of the squares of its digits.

92. The units digit of a 2-digit number is 1 more than its tens digit. If the digits are reversed, the product of this new number and the original number is 736.

93. A wheel makes a certain number of revolutions in traveling a distance of 1 mile. If the circumference of the wheel were 2 feet less, the wheel would make 88 more revolutions in traveling a mile. What is the circumference of the wheel?

94. A rectangular piece of tin is 6 inches longer than it is wide. An open box is made by cutting a 3-inch square from each corner and turning up the sides and ends. What are the dimensions of the box, if it holds 405 cubic inches?

95. A grocer paid $180 for some potatoes. He sold them at $1 a bushel more than he paid for them. When he still had 20 bushels left, he had already received $70 more than the potatoes had cost him. How many bushels had he bought?

96. Before taking his final examination in algebra Joe had an average grade of 55 per cent. The passing average was 60 per cent. If the final examination counts one-fourth and the other grades three-fourths, what is the lowest grade Joe may get on the final examination and still pass the course?

18.4 Equations and Graphs

We have defined an equation as an indicated equality between two functions. In solving an equation we are seeking the values of the variable which make the function on the left of the equals sign have the same value as the function on the right. For instance, for $2x + 3 = x + 5$ the value 2 for x has associated with it the value 7 for both the left-hand and right-hand functions. If we think of the graphs of these functions, $f(x) = 2x + 3$ and $\Phi(x) = x + 5$, at the point at which they intersect, the functions will have the same value. This will be the point (2, 7). The problem of solving an equation may be looked upon as seeking the x-coordinate of the point (or points) at which the graphs of the two functions intersect.

If we have an equation which is equivalent to the original equation, then the intersection of the graphs of the functions of this equation must have the same x-coordinate as the intersection of the graphs of the functions of the original equation. In particular, if we obtain an equivalent equation of the form $F(x) = 0$, then the solution of the original equation must be the x-coordinate of the point (or points) where the graph of $F(x)$ crosses the x-axis.

EXAMPLES 18.4.1

1. Determine from a graph the solution of $2x + 3 = x + 5$. The graphs of the functions $f(x) = 2x + 3$ and $\Phi(x) = x + 5$ intersect at (2, 7). An equivalent equation is $x - 2 = 0$. The graph of the function $F(x) = x - 2$ crosses the x-axis at (2, 0).

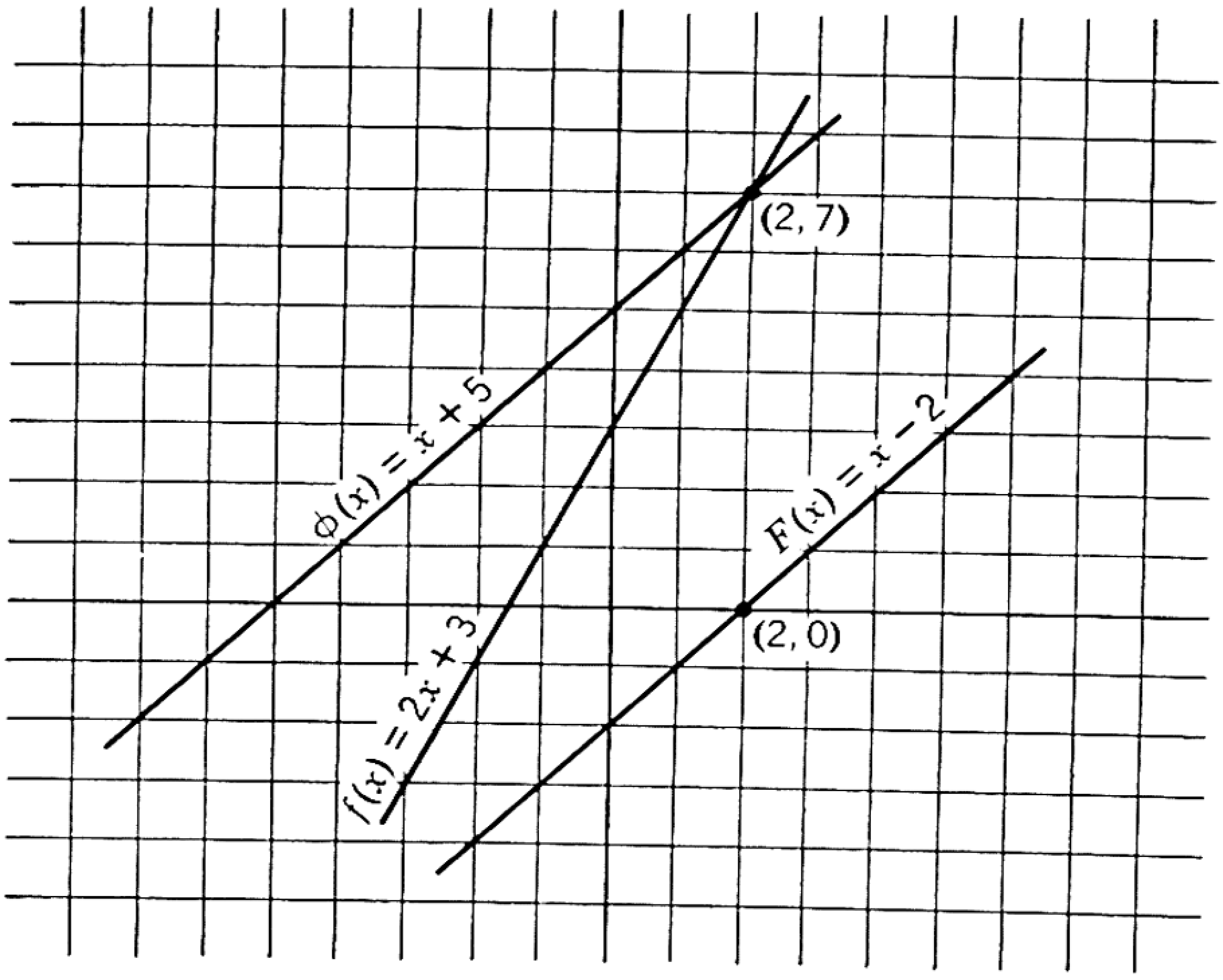


FIG. 18.4.2.

2. Determine from a graph the solution of $x^2 - 8 = 2x$. The graphs of the functions $f(x) = x^2 - 8$ and $\Phi(x) = 2x$ intersect at (4, 8) and (−2, −4). The solutions of the given equation are 4 and −2. An equivalent equation is $x^2 - 2x - 8 = 0$. The graph of $F(x) = x^2 - 2x - 8$ crosses the x-axis at (4, 0) and (−2, 0).

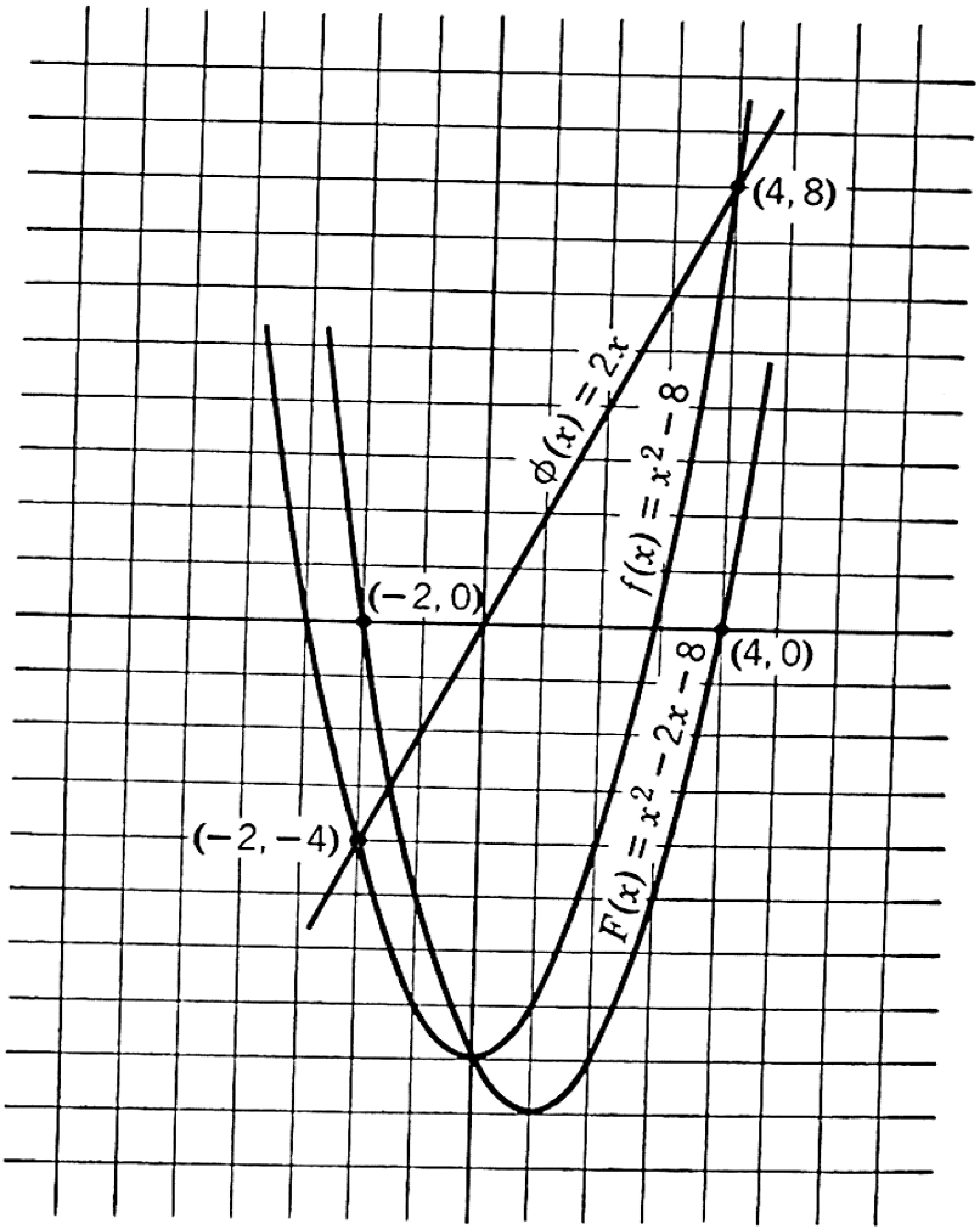


FIG. 18.4.3.

3. Determine from a graph the solution of $x^2 - 2x - 1 = 0$. The graph of the function $F(x) = x^2 - 2x - 1$ crosses the x-axis at approximately (2.4, 0) and (−0.4, 0). If we substitute these values for x in our equation we find that 2.4 and −0.4 are not quite solutions.

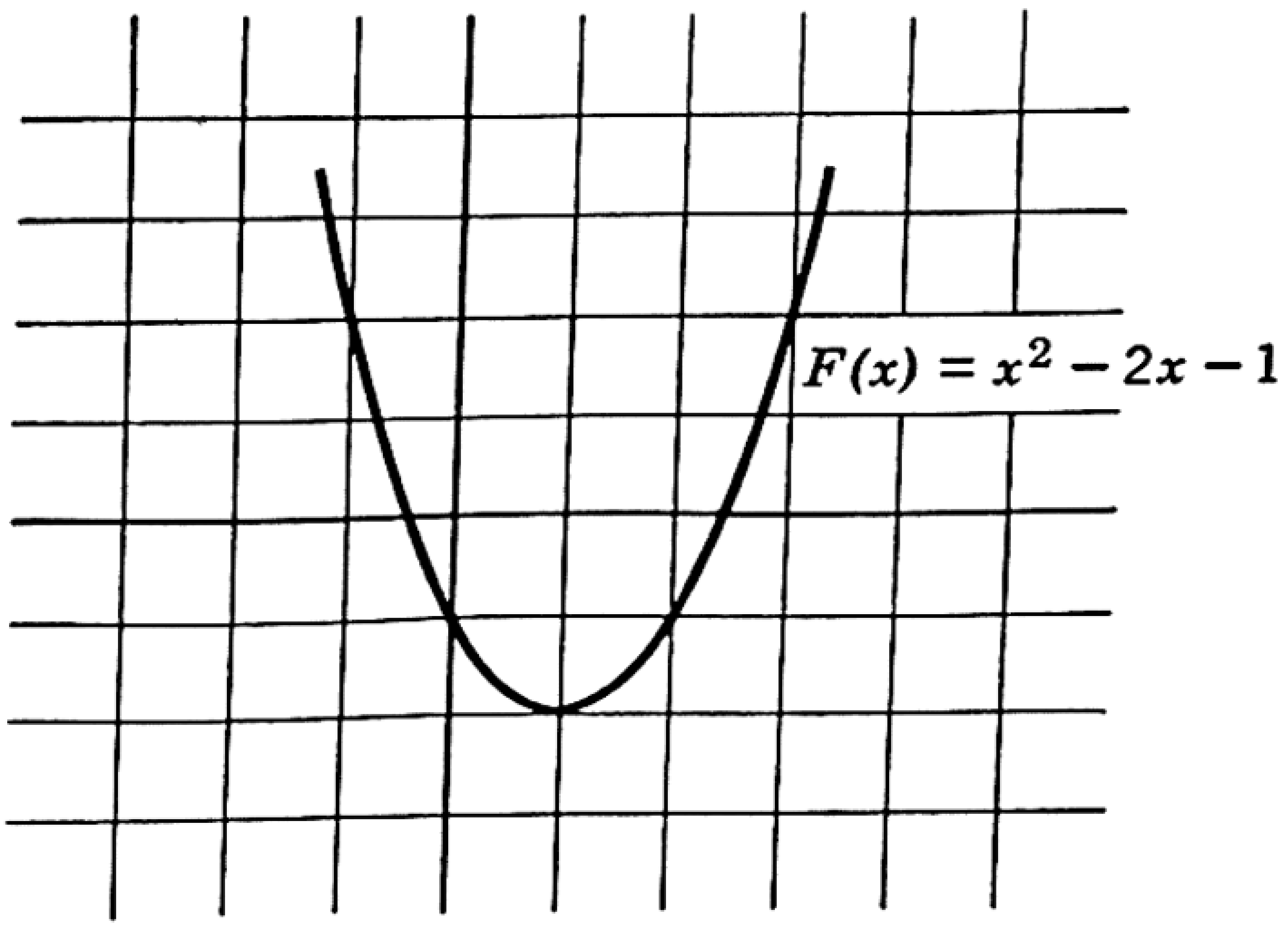


FIG. 18.4.4.

EXERCISE 18.4.5

By means of graphs find the solutions (or approximate solutions) of the given equations. Also graph an equivalent equation of the form $F(x) = 0$.

1. $3x - 4 = x$.
2. $3x + 4 = x$.
3. $3x - 3 = x + 4$.
4. $3x + 3 = x - 4$.
5. $x^2 = x + 2$.
6. $2x^2 + 6 = -x$.
7. $6x^2 - 25 = 5x$.
8. $2x^2 + x - 5 = x^2 - x - 2$.
9. $x^2 - 4x + 2 = 0$.
10. $x^2 - 3 = 0$.
11. $x^2 + 2x - 3 = x^2 - 2x + 5$.
12. $x^2 + x = x^2 + 3x - 4$.
13. $x^2 = -x - 2$.
14. $-x^2 = 2 - x$.

18.5 Simultaneous Equations

We have seen that when we have the graphs of two functions drawn on the same axes the x-coordinate of a point of intersection is a solution of the equation obtained when the functions are set equal to each other. The y-coordinate of such a point or the values of the functions at such a point are, of course, equal. If we now consider the two equations $y = f(x)$ and $y = \Phi(x)$, the coordinates of a point of intersection of their graphs give us a pair of numbers, an x and a y, which make both equations true statements simultaneously. A pair of such equations are called *simultaneous equations.* The pairs of values of x and y (if any) which make both equations true statements are called solutions of the simultaneous equations. A solution of simultaneous equations then consists of a number-pair (x, y), or geometrically it is a point of intersection of the graphs of the equations. The problem of solving an equation such as that of Problem 3 above is the same as the problem of finding the value of x such that the values of y of the equations $y = 3x - 3$ and $y = x + 4$ are equal.

Frequently simultaneous equations are not of the forms $y = f(x)$ and $y = \Phi(x)$. For instance, we may have

$$2x - 3y = 5$$

and

$$2x + 3y = 7.$$

These may be changed to equivalent simultaneous equations by using 18.2.3. We then should have

$$y = \frac{2x - 5}{3}$$

and

$$y = \frac{-2x + 7}{3}.$$

We call this "solving for y in terms of x." To complete the solution we equate these two functions of x and proceed as before.

Another method of solving these equations is to solve for y in terms of x in just one of the equations and then substitute the resulting function of x for y in the other equation. This again gives us an equation in x alone. For this problem we should have either

$$2x + 3\left(\frac{2x - 5}{3}\right) = 7$$

or

$$2x - 3\left(\frac{-2x + 7}{3}\right) = 5.$$

A third, and often the most efficient, way is to say: Suppose there are two numbers x and y such that

$$(a) \qquad 2x - 3y = 5$$

and at the same time

$$2x + 3y = 7.$$

Then for these numbers x and y

$$(b) \qquad 4x \qquad = 12.$$

(We added equals to equals.) This gives us $x = 3$, and we may now find y by substituting this value of x in either of the equations. Or we may find y by "eliminating x."

EXERCISE 18.5.1

Solve for x and y (do the first four by two methods):

1. $x + 2y = 3,$
   $x - 2y = 0.$

2. $2x + 5y = 2,$
   $x - 5y = -4.$

3. $3x + 4y = 7,$
   $3x + 2y = 1.$
4. $2x + 3y = -3,$
   $3x + 2y = 2.$
5. $x + 5 = 2y,$
   $y - 3 = 2x.$
6. $x + y = -8,$
   $2x + y = 4.$
7. $3x + 3y = 14,$
   $2x - 2y = 14.$
8. $5x - 7y = 17,$
   $4x + 9y = -2.$
9. $11x - 12y = 11,$
   $11x + 13y = 11.$
10. $2x - 23y = -5,$
    $4x - 5y = 0.$
11. $x + y = 3,$
    $2x + 2y = 5.$
12. $x - y = 2,$
    $3x - 3y = 6.$
13. $x - y = -1,$
    $xy = 2.$
14. $x - y = 2,$
    $xy - y = 30.$
15. $x + y = 3,$
    $x^2 + y^2 = 17.$
16. $\frac{x}{y} + \frac{y}{x} = \frac{5}{2},$
    $x + 2y = 4.$

For those verbal problems which we have had thus far we have used just one "unknown." If there was more than one quantity to be determined, before we could set up the equation we found these other quantities in terms of the one to which we had assigned a letter name.

Frequently it is simpler to use two (or more) "unknowns." Many of the problems we have already solved might have been done this way. For instance, Example 5 of 3.1.1 may be solved by letting x be the son's present age, y the father's present age, and then

$$y = x + 24$$

and

$$y + 10 = 2(x + 10).$$

Solving these simultaneous equations by substituting $x + 24$ for y in the second equation, we obtain $(x + 24) + 10 = 2(x + 10)$ as before.

Exercise 18.5.2

Set up a pair of simultaneous equations and solve them:

1. Find two numbers such that twice the first added to three times the second is 29 and twice the second subtracted from three times the first is 11.

2. Two numbers are such that twice the second added to the first gives a sum 17 and the second added to three times the first gives a sum of 16.

3. Find two numbers such that half the first added to a third of the second is 9 and a quarter of the first added to a fifth of the second is 5.

4. A number is a third of another, and their sum is 60.

5. A 2-digit number equals four times the sum of its digits; if we add 18 to the number, the sum has the same digits as the original number but in reverse order.

6. The sum of three numbers is 72. One-half of the first, one-third the second, and one-fourth the third are all equal.

7. A farmer bought 100 acres of land for $2450. For part of the land he paid $20 an acre, and for the rest he paid $30 an acre. How many acres were in each part? *

8. A laborer was engaged for 40 days on these conditions: for every day he worked he would receive 80 cents, but for every day he was idle he would forfeit 32 cents. At the end of the time he was entitled to $15.20. How many days did he work, and how many days was he idle? *

9. If the length of a rectangle is decreased by 3 feet and its width is increased by 2 feet, its area will be increased by 2 square feet; if its length is decreased by 2 feet and its width is increased by 1 foot, its area will not be changed.

10. Two objects of unknown weights balance when one is placed 16 inches and the other 24 inches from a fulcrum. When the positions of the objects are reversed, it is found that 25 pounds must be added on the side of 16-inch leverage before a balance is obtained. What are the weights of the objects?

11. The product of two numbers is 40 more than the smaller, and their difference is 4.

12. The sum of two numbers is 13, and the sum of their squares is 89.

13. The hypotenuse of a right triangle is 29, and the sum of the legs is 41.

14. The difference between the hypotenuse and one leg of a right triangle is 72 and the other leg is 84.

---

* From James B. Dodd: *Elementary and Practical Algebra,* copyright 1852.

When we want to use three "unknowns" in a problem, we obtain, if the problem is solvable, three equations. A solution of three equations in three unknowns, say x, y, and z, will consist of a number-triple which we may write (x, y, z). In order to find the solution we may use methods similar to those used for two equations in two unknowns.

Our first objective will be to obtain from the three equations in three unknowns a pair of equations in two unknowns. When we have solved these two equations for two of the unknowns, then the third unknown may be found by substituting the values of the two in one of the original equations.

Example 18.5.3

Solve the equations

$$x + y + 2z = 0,$$
$$2x - y - z = 2,$$
$$x + 2y - z = 4.$$

Method 1. Solve the first and second equations for z in terms of x and y and set the results equal to each other. We obtain

$$\frac{-x - y}{2} = 2x - y - 2.$$

Now treating the second and third the same way, we obtain

$$2x - y - 2 = x + 2y - 4.^{*}$$

These last two equations may now be solved for x and y. We find $x = 1$ and $y = 1$. Substituting in the first equation, we find $z = -1$.

Method 2. Solve one of the equations for x in terms of y and z. If we use the first equation we obtain $x = -y - 2z$. We substitute this for x in the other two equations and proceed as before.

Method 3. Using the first two equations, we eliminate the y by adding the equations, obtaining

$$3x + z = 2.$$

Using the second and third (or first and third) and again eliminating y, we obtain

$$5x - 3z = 8.$$

---

* Or we might have used the first and third.

Exercise 18.5.4

Solve for x, y, and z. Do the first and second by three methods.

1. $x + y + z = 4,$
$x - y + z = 2,$
$x + y - z = 2.$

2. $2x + y + z = 5,$
$x + 2y + z = 0,$
$x + y + 2z = 3.$

3. $x + z = 2,$
$x + y = -5,$
$y + z = 1.$

4. $x + y + z = 0,$
$-x + 3y + z = 1,$
$2x - y - 2z = 3.$

5. A cistern holds 820 gallons. It is filled by three pipes in 20 minutes. The first pipe can convey 10 gallons more per minute than the third pipe, and the second pipe can convey 5 gallons less per minute than the third pipe. How much flows through each pipe in a minute?

6. A tank holds 240 gallons. It is filled by three pipes. If water is flowing through all three pipes, it takes 20 minutes to fill it; if it is flowing through only the first two pipes it takes 30 minutes; if it is flowing through the last two only, it takes 40 minutes. How many gallons did each pipe discharge per minute?

7. The sum of three numbers is 6. The second equals twice the difference between the first and third, and the first equals twice the sum of the second and third.

8. A man has three sums of money invested, one at 2 per cent, one at 3 per cent, and one at 3½ per cent. His total income from the three investments is $246 per year. The first investment yields $14 more per year than the sum of the other two. If all the money were invested at 2½ per cent, his income would be $4 per year more than it is now. How much is invested at each rate?

19

The Real Number System

19.1 Irrational Numbers

If we raise the question, What integer added to itself gives the sum 6? we readily obtain the answer 3. We might rephrase the question to read, What is the solution of the equation $2x = 6$? If we ask the question, What integer added to itself gives us the sum 5? or What is the solution of the equation $2x = 5$? we find that there is no (integer) solution. For such reasons we invented the rational numbers.

Suppose now that we ask, What is the solution of the equation $x^2 = 9$ or what number *multiplied* by itself gives the product 9? we find that there are two answers, 3 and -3. But, if we raise the question, What is the solution of the equation $x^2 = 2$ or what number multiplied by itself gives the product 2? we find ourselves in a difficulty.

Certainly the positive solution must be greater than 1, since $1^2 = 1$, and at the same time it must be less than 2, since $2^2 = 4$. We may indicate these facts by writing

$$1 < x < 2.$$

Suppose that we guess that the solution is 1.5. If we square this number, we obtain 2.25, so that our guess was too large. We try 1.4 and find it is too small, since $(1.4)^2 = 1.96$. However, we may state that $1.4 < x < 1.5$.

If we try 1.45, we find it also is too large, since $(1.45)^2 = 2.1025$. Trying 1.44, 1.43, 1.42, and 1.41 in order, we find that 1.42 is too large and 1.41 is too small,

$$1.41 < x < 1.42.$$

If we continue our search by trying numbers between 1.41 and 1.42, such as 1.413, we find that

$$1.414 < x < 1.415,$$

and with some more work we find

$$1.4142 < x < 1.4143$$

and

$$1.41421 < x < 1.41422$$

and

$$1.414213 < x < 1.414214$$

and

$$1.4142135 < x < 1.4142136.$$

We seem to be on a difficult quest. The only numbers that we know about at this stage are rational numbers, and, if this number that we are seeking is rational, it must be, in the decimal notation, a repeating or terminating decimal. (Perhaps we have not been patient or persevering enough. When we sought $1/23$ as a repeating decimal, we saw how persevering we must be at times.)

So that you will not spend the rest of your lives trying to find this repeating or terminating decimal whose square is 2, we stop you at this point by making the astonishing statement: There is *no* repeating or terminating decimal whose square is 2. *The number whose square is 2 is not a rational number.* If we were to try as many numbers from the display of 12.4 as our patience would allow us, we should never find one whose square is 2! *

We shall now prove that there is no rational number x such that $x^2 = 2$. In proving this we shall make use of the fact, which we have mentioned before (11.3.3), that every composite positive integer (natural number) can be expressed as the product of prime numbers in one and only one way. Also, it is obvious that, if a is a natural number, then a^2 has the same prime factors as a but each occurring twice as many times as in a; each prime factor must occur an *even* number of times in a^2. For instance, $24 = 2^3 \cdot 3$, but $24^2 = 2^6 \cdot 3^2$.

---

* The first people to be astonished by this were Pythagoras and his followers (about 500 B.C.). The question arose with them when the Pythagorean theorem was applied to the diagonal of a square of side 1 unit. If we call the diagonal x units, then $x^2 = 1^2 + 1^2 = 2$.

Suppose then that $x^2 = 2$ and $x = a/b$, where a and b are integers. Then

$$\left(\frac{a}{b}\right)^2 = 2 \quad \text{or} \quad \frac{a^2}{b^2} = 2;$$

and

$$a^2 = 2b^2.$$

Consider the number b^2. If it has the factor 2, it has the factor 2 an even number of times and $2b^2$ has the factor 2 an odd number of times. But this says that a^2, the square of an integer, has the factor 2 an odd number of times, which is impossible.

We see that, if $(a/b)^2 = 2$, where a and b are integers, is a true statement, then $a^2 = 2b^2$ must be a true statement. But we know that the second statement is false, and therefore the first must be false and there is no pair of integers a and b such that $(a/b)^2 = 2$; or the number whose square is 2 cannot be a rational number.

19.1.1 Definition: A number which cannot be written as the quotient of two integers is called an *irrational* number.*

We may say that 1.4142 is a 4-decimal-place rational approximation to the number whose square is 2.

Exercise 19.1.2

Using the method we used above, find a 3-decimal-place rational approximation to the positive solution of:

1. $x^2 = 3$.
2. $x^2 = 5$.
3. $x^3 = 2$.
4. $x^4 = 2$.

5. Show that the positive solution of the equation of Problem 1 is irrational.

---

* It can be proved that the solutions of any equation of the form $x^n = c$, where c is an integer but not the nth power of an integer, are irrational.

Those irrational numbers that we obtain as roots of equations of the form $P(x) = 0$, such as $x^2 - 2 = 0$, where $P(x)$ is a polynomial with rational coefficients, are called algebraic irrationals. However, there is another kind of irrational number called a transcendental number. A transcendental number is not the root of any equation of the form $P(x) = 0$, where $P(x)$ is a polynomial with rational coefficients. The ratio of the circumference of a circle to its diameter is a transcendental number. This ratio is usually indicated by the Greek letter π.

6. Show that the solution of the equation of Problem 3 is irrational.

7. Show that, if $x^2 = 2$, then $x + 1$ is irrational. (Hint: Suppose that $x + 1 = y$ is rational. This leads to a contradiction.)

8. Show that, if x is irrational and r is rational, then (*a*) $x + r$ is irrational; (*b*) $x - r$ is irrational; (*c*) $r \cdot x$ is irrational; (*d*) r/x is irrational; * (*e*) x/r is irrational.

9. Show that the irrational numbers are not closed under the operation of addition. (Hint: x and $-x$ are both irrational, if $x^2 = 2$; also $0.1011011101111\cdots$ and $0.0100100010000\cdots$ are both irrational.)

10. Show that the irrational numbers are not closed under the operation of (*a*) subtraction, (*b*) multiplication, (*c*) division.

19.2 Radicals

The symbol that we use for the positive number whose square is 2 is $\sqrt{2}$; the symbol that we use for the negative number whose square is 2 is $-\sqrt{2}$. These are read, respectively, "the positive (or principal) square root of 2" and "the negative square root of 2." We then have $(\sqrt{2})^2 = 2$ and $(-\sqrt{2})^2 = 2$.

19.2.1 DEFINITION: If there exists a number x such that $x^n = a$, then x is said to be an nth root of a. If n is 2, then x is a *square root* of a, and, if n is 3, then x is a *cube root* of a.

19.2.2 DEFINITION: If there is a positive number x such that $x^n = a$, then this number is called the *principal* (or positive) nth root of a. If there is no positive number x such that $x^n = a$ but there is a negative number y such that $y^n = a$, then this number y is the principal nth root of a.

We indicate the principal nth root of a by the symbol $\sqrt[n]{a}$. If n is 2, we write simply $\sqrt{a}$. The symbol $\sqrt[n]{a}$ is called a radical; n is the index of the radical, and a is the radicand.

---

* The symbol a/b will now be identified with $a \div b$ where a and b are any numbers whatsoever.

Originally we defined a fraction as a symbol a/b where a and b are natural numbers. Later we called this a positive rational number. Hereafter a fraction will be any symbol of the form a/b where a and b are any numbers whatsoever, except $b \neq 0$.

Examples 19.2.3

1. $\sqrt{4} = 2$ and $-\sqrt{4} = -2$.
2. $\sqrt[3]{8} = 2$ is the principal cube root of 8.
3. $\sqrt{2}$ is the principal (or positive) square root of 2.
4. $-\sqrt{2}$ is the negative square root of 2.
5. $\sqrt[3]{-8} = -2$ is the principal cube root of -8.
6. $\sqrt[4]{81} = 3$ and $-\sqrt[4]{81} = -3$.

Exercise 19.2.4

Read each of the following:

1. $\sqrt{9} = 3$.
2. $-\sqrt{9} = -3$.
3. $\sqrt[3]{27} = 3$.
4. $\sqrt[3]{-27} = -3$.
5. $-\sqrt[3]{27} = -3$.
6. $-\sqrt[3]{-27} = -(-3) = 3$.
7. $\sqrt[5]{32} = 2$.
8. $\sqrt[5]{-32} = -2$.
9. $(\sqrt{2}\,)^2 = 2$.
10. $(-\sqrt{2}\,)^2 = 2$.
11. $(\sqrt[3]{4}\,)^3 = 4$.
12. $(\sqrt[n]{a}\,)^n = a$.
13. If $a > 0$, then $$\begin{aligned}(-\sqrt[n]{a}\,)^n &= a \text{ if } n \text{ is even}\\ &= -a \text{ if } n \text{ is odd.}\end{aligned}$$
14. $1/\sqrt{2} = \sqrt{2}/2$.
15. $(\sqrt{3}\,)^3 = \sqrt{3^3} = 3\sqrt{3}$.
16. $\sqrt[3]{2^2} = (\sqrt[3]{2}\,)^2$.
17. $\sqrt{8} + \sqrt{50} = 2\sqrt{2} + 5\sqrt{2} = 7\sqrt{2}$.
18. $\sqrt{2}\cdot\sqrt{8} = \sqrt{16} = 4$.
19. $$\begin{aligned}\sqrt{(a-b)^2} &= a - b \text{ if } a > b\\ &= b - a \text{ if } a < b\\ &= 0 \text{ if } a = b.\end{aligned}$$

Evaluate:

20. $\sqrt{16} - \sqrt[4]{16}$.
21. $\sqrt[3]{-8} + \sqrt[3]{8}$.
22. $\sqrt{\frac{25}{4}} + \sqrt[3]{\frac{27}{8}}$.
23. $\sqrt{1 - \left(\frac{\sqrt{3}}{2}\right)^2}$.
24. $(\sqrt{3}\,)^4 - \sqrt{81}$.
25. $\sqrt{\frac{1 - \frac{1}{2}}{2}}$.
26. $\sqrt{\frac{\frac{1}{3} + 1}{3}}$.
27. $\sqrt{1 - (\frac{3}{5})^2}$.
28. $\sqrt{36}$.
29. $\sqrt{9}\cdot\sqrt{4}$.

19.3 Tables

At times it is necessary to find an approximation to a root of a number. The degree of accuracy that we need, the number of digits, depends upon the use to which we want to put the approximation. If the approximation that we wish to find is to be used in fitting a brace to a fence, then it would be silly to find the approximation to more than 2 decimal places when the dimensions are in feet. But, if the approximation is to be used in determining the diagonal of a rectangular machine part, it may be necessary to find it correct to 4 decimal places (when the sides are measured in inches).

We may find the needed approximations by the method previously described for finding $\sqrt{2}$. But square and cube roots are needed so frequently that tables have been constructed from which we may readily obtain the approximations. Part of such a table (correct to 3 decimal places) is given on page 233. (Which numbers in this table are exact square roots or cube roots of the numbers in the first column?) [35]

Exercise 19.3.1

Find 3-decimal-place approximations from the table:

1. $\sqrt{7}$.
2. $\sqrt{21}$.
3. $\sqrt{28}$.
4. $2\sqrt{7}$.
5. $\sqrt{48}$.
6. $4\sqrt{3}$.
7. $\sqrt[3]{40}$.
8. $2\sqrt[3]{5}$.
9. $10\sqrt{5}$.
10. $\sqrt{500}$.
11. $\sqrt{50}$.
12. $\sqrt[3]{2000}$.
13. $10\sqrt[3]{2}$.
14. $\sqrt[3]{200}$.
15. $\sqrt{3} + \sqrt{7}$.
16. $\sqrt{11} - \sqrt{7}$.
17. $\sqrt{2} \cdot \sqrt{3}$.
18. $\sqrt{6}$.
19. $\sqrt{2}/2$.
20. $1/\sqrt{2}$.
21. $\sqrt{13}/3$.
22. $\dfrac{\sqrt{3} + \sqrt{2}}{3}$.
23. $\dfrac{\sqrt{3} + \sqrt{2}}{\sqrt{3}}$.
24. $\dfrac{3 + \sqrt{6}}{3}$.
25. Graph $y = \sqrt{x}$.
26. Graph $y = -\sqrt{x}$.
27. Graph $y = \sqrt[3]{x}$.
28. Graph $y = \sqrt[3]{-x}$.
29. Graph $y = \sqrt{10 - x^2}$.
30. Graph $y = -\sqrt{10 - x^2}$

Square Roots and Cube Roots

n	$\sqrt{n}$	$\sqrt{10n}$	$\sqrt[3]{n}$	$\sqrt[3]{10n}$	$\sqrt[3]{100n}$
1	1.000	3.162	1.000	2.154	4.642
2	1.414	4.472	1.260	2.714	5.848
3	1.732	5.477	1.442	3.107	6.694
4	2.000	6.325	1.587	3.420	7.368
5	2.236	7.071	1.710	3.684	7.937
6	2.449	7.746	1.817	3.915	8.434
7	2.646	8.367	1.913	4.121	8.879
8	2.828	8.944	2.000	4.309	9.283
9	3.000	9.487	2.080	4.481	9.655
10	3.162	10.000	2.154	4.642	10.000
11	3.317	10.488	2.224	4.791	10.323
12	3.464	10.954	2.289	4.932	10.627
13	3.606	11.402	2.351	5.066	10.914
14	3.742	11.832	2.410	5.192	11.187
15	3.873	12.247	2.466	5.313	11.447
16	4.000	12.649	2.520	5.429	11.696
17	4.123	13.038	2.571	5.540	11.935
18	4.243	13.416	2.621	5.646	12.164
19	4.359	13.784	2.668	5.749	12.386
20	4.472	14.142	2.714	5.848	12.599
21	4.583	14.491	2.759	5.944	12.806
22	4.690	14.832	2.802	6.037	13.006
23	4.796	15.166	2.844	6.127	13.200
24	4.899	15.492	2.884	6.214	13.389
25	5.000	15.811	2.924	6.300	13.572
26	5.099	16.125	2.962	6.383	13.751
27	5.196	16.432	3.000	6.463	13.925
28	5.292	16.733	3.037	6.542	14.095
29	5.385	17.029	3.072	6.619	14.260
30	5.477	17.321	3.107	6.694	14.422
31	5.568	17.607	3.141	6.768	14.581
32	5.657	17.889	3.175	6.840	14.736
33	5.745	18.166	3.208	6.910	14.888
34	5.831	18.439	3.240	6.980	15.037
35	5.916	18.708	3.271	7.047	15.183
36	6.000	18.974	3.302	7.114	15.326
37	6.083	19.235	3.332	7.179	15.467
38	6.164	19.494	3.362	7.243	15.605
39	6.245	19.748	3.391	7.306	15.741
40	6.325	20.000	3.420	7.368	15.874
41	6.403	20.248	3.448	7.429	16.005
42	6.481	20.494	3.476	7.489	16.134
43	6.557	20.736	3.503	7.548	16.261
44	6.633	20.976	3.530	7.606	16.386
45	6.708	21.213	3.557	7.663	16.510
46	6.782	21.448	3.583	7.719	16.631
47	6.856	21.679	3.609	7.775	16.751
48	6.928	21.909	3.634	7.830	16.869
49	7.000	22.136	3.659	7.884	16.985
50	7.071	22.361	3.684	7.937	17.100

19.4 The Real Number System

The set composed of all the numbers that we have discussed thus far is said to form the real number system.* It is made up of the positive and negative rational numbers, the positive and negative irrational numbers and zero.

The system of real numbers is closed under the operations of addition, subtraction, multiplication, and division. The members of the real number system follow all the rules which we have formulated for our previous systems. The fundamental rules are:

If a, b, and c are real numbers, then

$$a + b = b + a,$$

$$a \cdot b = b \cdot a,$$

$$a + (b + c) = (a + b) + c,$$

$$a(b \cdot c) = (a \cdot b)c,$$

$$a(b + c) = ab + ac.$$

---

* The word "real" as used in "real number" is unfortunately misleading. We have a tendency to associate with a word its ordinary meaning, in this case something like "genuine." We meet other words in mathematics (and elsewhere) which have technical meanings which have little or nothing to do with their ordinary meanings. Another example is "odd number." This is, of course, simply an integer which is not divisible by 2. To many people "odd" means "strange or unusual." But we have grown so accustomed to this technical meaning of "odd" that we usually do not give it a second thought; when we say "odd number," we mean a number not divisible by 2; when we say "odd person," we are using the word in its other sense.

It is amusing that an excellent little book by Herbert McKay has the title *Odd Numbers* (The Macmillan Co., 1944). Here the word "odd" is being used with the meaning "unusual," which is a rather confusing situation!

Some students, when not on their toes, speak of even numbers and *uneven* numbers. By the latter we suppose that they mean odd numbers. "Uneven numbers" is a fine combination of words; it happens that it is not the technical term used in contrast with even in speaking of numbers. Of course, it might have been used in this sense, and then "uneven numbers" would be what we call "odd numbers."

"What's in a name? that which we call a rose
By any other name would smell as sweet."

However, it is important that we try to use words which mean the same to all. In the present case a real number is simply a number which satisfies 19.4.1.

Also, in particular, they follow the laws of signs for multiplication: The product of two real numbers of like sign is a positive number, and the product of two real numbers of unlike signs is a negative number; the product of any number and zero is zero. In a very important sense which we shall appreciate later this rule characterizes real numbers:

19.4.1 *A real number is a number whose square is either positive or zero.*

19.5 Rational Exponents and Radicals

We have given meaning to symbols of the form a^n where n is an integer, positive, negative, or zero. For a positive exponent n, a^n means the product of n a's. For a negative exponent $-n$ we defined $a^{-n} = 1/a^n$, and for n zero we defined $a^0 = 1$ if $a \neq 0$.

It has been found to be of considerable advantage to extend the meaning of a^n so as to give meaning to symbols such as $2^{1/2}$ and $5^{-2/3}$. Whatever meaning we give to these symbols, we shall want them to obey the same rules as those obeyed by integer exponents. We should not like it if $(a^m)^n = a^{mn}$ for integer exponents but something else for fractional exponents.

Since $a^{1/n}$, if it is to mean anything, should obey the laws of exponents, we should have $(a^{1/n})^n = a$. We defined a solution of the equation $x^n = a$ as an nth root of a and indicated the principal nth root by $\sqrt[n]{a}$. By definition then $(\sqrt[n]{a})^n = a$. We, therefore, state the

19.5.1 Definition: $a^{1/n} = \sqrt[n]{a}$ (the principal nth root of a).

Applying our rules to the symbol $a^{m/n}$, we should have

$$a^{m/n} = (a^m)^{1/n} = (a^{1/n})^m$$

and therefore state the

19.5.2 Definition: $a^{m/n} = \sqrt[n]{a^m} = (\sqrt[n]{a}\,)^m$.

Obviously, if we are to extend our definition to negative fractional exponents, we should have

$$a^{-m/n} = \frac{1}{a^{m/n}}.$$

It can be shown that all our laws of exponents apply to operations with these new kinds of exponents.*

EXAMPLES 19.5.3

1. $a^{1/2} = \sqrt{a}$.
2. $5^{2/3} = \sqrt[3]{5^2} = \sqrt[3]{25} = (\sqrt[3]{5}\,)^2$.
3. $\sqrt{a + b} = (a + b)^{1/2} \neq a^{1/2} + b^{1/2}$.
4. $\sqrt[3]{x^3 + y^3} = (x^3 + y^3)^{1/3} \neq x + y$.

EXERCISE 19.5.4

Express in equivalent radical form (or without radicals or fractional exponents if possible):

1. $2^{1/2}$, $3^{2/3}$, $4^{1/2}$, $27^{1/3}$, $17^{5/2}$.
2. $2^{-1/2}$, $3^{-2/3}$, $4^{-1/2}$, $27^{-1/3}$, $17^{-5/2}$.
3. $(27^{1/2})^2$, $(3 \cdot 3^{1/2})^2$, $(16^{1/3})^3$, $(2 \cdot 2^{1/3})^3$.
4. $2^{2/3}$, $2^{3/2}$, $2^{-2/3}$, $2^{-3/2}$.
5. $(3^2)^{1/2}$, $(5^3)^{1/3}$, $(2^3)^{2/3}$, $(3^4)^{3/2}$.
6. $(a^2 + b^2)^{1/2}$, $(a^2 - b^2)^{1/2}$.
7. $(a + b)^{1/2}$, $(a - b)^{1/2}$.
8. $[4(x^2 + y^2)]^{1/2}$, $[8(x^3 - y^3)]^{1/3}$.
9. $10^{0.3}$, $10^{0.5}$, $10^{0.301}$, $10^{0.477}$, $10^{0.8451}$.
10. $10^{-0.7}$, $10^{-0.5}$, $10^{-0.699}$, $10^{-0.523}$, $10^{-0.1549}$.

Express in equivalent exponent form:

11. $\sqrt{c}$.
12. $\sqrt[4]{a^2}$.
13. $\sqrt{t^3}$.
14. $\sqrt[3]{s^2}$.
15. $3 \cdot \sqrt{x}$.
16. $\sqrt{3x}$.
17. $\sqrt{9x}$.
18. $\sqrt{p^2 + q^2}$.
19. $\sqrt{(p + q)^2}$.
20. $\sqrt[3]{a^3 - b^3}$.
21. $\sqrt{9(x^2 - y^2)}$.
22. $a \cdot \sqrt{a}$.
23. $b^2 \cdot \sqrt[3]{b}$.
24. $a \cdot \sqrt{ab}$.
25. $b \cdot \sqrt{ab}$.
26. $ab \cdot \sqrt{ab}$.
27. $(\sqrt{x}\,)^4$.
28. $(\sqrt{xy}\,)^4$.
29. $(\sqrt{x^5y^3}\,)^4$.
30. $(\sqrt[3]{x^5y^3}\,)^6$.

---

* At the present time we must restrict the base a in $a^{m/n} = \sqrt[n]{a^m} = (\sqrt[n]{a}\,)^m$ to non-negative numbers if n is even. At the present time, for instance, $(-1)^{1/2} = \sqrt{-1}$ is without meaning. If letters appear in the base of any problem, we shall assume, if necessary, that the letter may take on only values which give the expression meaning.

Perform the indicated operations (write answers with positive exponents if any are necessary):

31. $x^{1/2} \cdot x^{2/3}$.
32. $a^{1/3} \cdot a^{2/3}$.
33. $c \cdot c^{1/2}$.
34. $4^{1/2} \cdot 4$.
35. $t^{2/5} \cdot t^{1/3}$.
36. $x^0 \cdot x^{5/4}$.
37. $b^{3/2} \div b^{1/2}$.
38. $b^{1/2} \div b^{3/2}$.
39. $c^{2/3} \div c^{1/2}$.
40. $r^{2/3} \cdot r^{-1/2}$.
41. $x^{1/2} \cdot x^{-2/3}$.
42. $x^{1/2} \div x^{-2/3}$.
43. $x^{-1/2} \div x^{2/3}$.
44. $a^{2/3} \cdot a^{1/2} \cdot a^{-1/3}$.
45. $b^{-2/3} \cdot b^{-1/2} \cdot b^{1/3}$.
46. $10^{0.3} \cdot 10^{0.7}$.
47. $10^{0.301} \cdot 10^{0.699}$.
48. $10^{0.477} \cdot 10^{0.523}$.
49. $10^{0.3} \div 10^{-0.7}$.
50. $10^{0.477} \div 10^{-0.523}$.

Evaluate:

51. $16^{1/2} - 16^{1/4}$.
52. $(-8)^{1/3} + 8^{1/3}$.
53. $(\frac{25}{4})^{1/2} + (\frac{27}{8})^{1/3}$.
54. $(3^{1/2})^4 - 81^{1/2}$.
55. $(\frac{1}{8})^{-1/3} + (\frac{1}{8})^{1/3}$.
56. $(\frac{1}{8})^{-1/3} \div (\frac{1}{8})^{1/3}$.
57. $(\frac{1}{9})^{3/2} \div (\frac{1}{9})^{-3/2}$.
58. $(58)^{1/2} \cdot (58)^{-1/2}$.
59. $125^{2/3} \cdot 36^{3/2}$.
60. $81^{3/2} \cdot 16^{-3/4}$.
61. $(0.09)^{1/2} - (0.25)^{1/2}$.
62. $(0.027)^{1/3} + (0.125)^{1/3}$.
63. $(0.16)^{-1/2} + (0.36)^{1/2}$.
64. $(0.008)^{2/3} + (0.008)^{-2/3}$.
65. $(0.04)^{0.5} + (0.09)^{1.5}$.
66. $(0.0016)^{0.75} - (0.04)^{1.5}$.
67. $9^{-1/2} + 16^{-1/2}$.
68. $(9 + 16)^{-1/2}$.
69. $\dfrac{25^{-1/2} + 16^{-1/2}}{25^{-1/2} - 16^{-1/2}}$.
70. $\dfrac{(25 - 9)^{-1/2}}{25^{-1/2} - 9^{-1/2}}$.

Since $a^{m/n} = (a^m)^{1/n} = (a^{1/n})^m = \sqrt[n]{a^m} = (\sqrt[n]{a}\,)^m$, we may evaluate an expression such as $\sqrt[4]{16^3} = (\sqrt[4]{16}\,)^3$ by at least two methods. One way would be to cube 16 and then find the fourth root of the result. This leaves us with the problem of finding $\sqrt[4]{4096}$. This is 8, as we can easily verify. On the other hand, if we use the right side of the equality, we have the problem of finding $\sqrt[4]{16}$ and cubing the result. This, of course, should and does give us 8. But this second method involves no number greater than 16.

EXERCISE 19.5.4 (continued)

71. $\sqrt[3]{27^2}$.
72. $\sqrt{25^3} - \sqrt[3]{125^2}$.
73. $16^{5/4} - 8^{5/3}$.
74. $81^{3/4} + 9^{5/2}$.
75. $\sqrt[4]{(0.0016)^3}$.
76. $\sqrt[3]{(0.027)^4}$.

Since $a^m \cdot a^n = a^{m+n}$ (when m and n are any rational numbers), we may write $c\sqrt{c} = c \cdot c^{1/2} = c^{3/2} = \sqrt{c^3}$. In general, any factor which is "outside" a radical may be placed "under" the radical if this factor is first raised to a power equal to the index of the radical. Also we may remove from under a radical any "perfect power" factor, a factor which may be written with an exponent equal to the index of the radical.

EXAMPLES 19.5.5

1. $3\sqrt{6} = \sqrt{3^2\cdot 6} = \sqrt{54}$.
2. $a\sqrt{b} = \sqrt{a^2b}$.
3. $2x^2\sqrt{x} = \sqrt{2^2(x^2)^2\cdot x} = \sqrt{4x^5}$.
4. $2t^2s\cdot\sqrt[3]{3ts^2} = \sqrt[3]{2^3t^6s^3\cdot 3ts^2} = \sqrt[3]{24t^7s^5}$.
5. $\sqrt{32} = \sqrt{16\cdot 2} = 4\sqrt{2}$.
6. $\sqrt[3]{24x^5y^7} = \sqrt[3]{2^3x^3(y^2)^3\cdot 3x^2y} = 2xy^2\sqrt[3]{3x^2y}$.

We shall say that a radical is in simplest form if there are no perfect-power factors under the radical sign. If we have $5\sqrt{2} + 3\sqrt{2}$, this may be written as $8\sqrt{2}$; but we may not combine $4\sqrt{2} + 7\sqrt{3}$ in this way.

EXERCISE 19.5.6

Write in simplest form:

1. $\sqrt{8}$.
2. $\sqrt{50}$.
3. $\sqrt{8} + \sqrt{50}$.
4. $\sqrt{48} - \sqrt{75}$.
5. $\sqrt{98} - \sqrt{12}$.
6. $3\sqrt{28} + 2\sqrt{63}$.
7. $5\sqrt{288} + 5\sqrt{300}$.
8. $2\sqrt[3]{16} + 5\sqrt[3]{54}$.
9. $10\sqrt[3]{500} - 10\sqrt[3]{250}$.
10. $5\sqrt{75} - 7\sqrt{108} + 6\sqrt{245}$.
11. $3\sqrt[3]{40} + 2\sqrt[3]{135} - 3\sqrt[3]{320}$.
12. $2\sqrt{50} + 3\sqrt{32} - 6\sqrt{75}$.
13. $\sqrt{9x^3}$.
14. $\sqrt{27x^3}$.
15. $\sqrt{5t^3s^5}$.
16. $\sqrt{45t^7s^8}$.
17. $\sqrt[3]{5t^3s^2}$.
18. $\sqrt[3]{45t^7s^8}$.
19. $3a\sqrt{4a^3b^2}$.
20. $5ab\sqrt{12a^6b^5c^4}$.

Write with no factor outside the radical:

21. $3\sqrt{2}$.
22. $3a\sqrt{2}$.
23. $2xy^2\sqrt{x}$.
24. $3t^3s^4\sqrt{5t}$.
25. $t\sqrt[3]{5s^2}$.
26. $2xy^2\sqrt[3]{3x^2y}$.

19.6 Multiplication and Division of Radicals

When we have the problem of multiplying or dividing radicals of the same index we may make use of the rules

$$\sqrt[n]{a} \cdot \sqrt[n]{b} = a^{1/n} \cdot b^{1/n} = (ab)^{1/n} = \sqrt[n]{ab}$$

and

$$\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \frac{a^{1/n}}{b^{1/n}} = \left(\frac{a}{b}\right)^{1/n} = \sqrt[n]{\frac{a}{b}}.$$

We see that, if the two radicals have the same index, we may simply multiply or divide the radicands, as indicated.

In the case of multiplication or division of radicals of different indices it is usually best to write them in the exponent form and then use the laws of exponents.

EXAMPLES 19.6.1

1. $\sqrt{3} \cdot \sqrt{5} = \sqrt{15}$.
2. $\sqrt{3} \cdot \sqrt{12} = \sqrt{36} = 6$ or $\sqrt{3} \cdot \sqrt{12} = \sqrt{3} \cdot 2 \cdot \sqrt{3} = 2 \cdot 3 = 6$.
3. $\sqrt{2} \cdot \sqrt[3]{2} = 2^{1/2} \cdot 2^{1/3} = 2^{5/6} = \sqrt[6]{2^5} = \sqrt[6]{32} = (\sqrt[6]{2}\,)^5$.
4. $\sqrt{2} \cdot \sqrt[3]{3} = 2^{1/2} \cdot 3^{1/3} = 2^{3/6} \cdot 3^{2/6} = (2^3 \cdot 3^2)^{1/6} = \sqrt[6]{2^3 \cdot 3^2} = \sqrt[6]{72}$.
5. $\dfrac{\sqrt{3}}{\sqrt{5}} = \sqrt{\dfrac{3}{5}} \cdot$
6. $\dfrac{\sqrt{12}}{\sqrt{3}} = \sqrt{\dfrac{12}{3}} = \sqrt{4} = 2$ or $\dfrac{\sqrt{12}}{\sqrt{3}} = \dfrac{2 \cdot \sqrt{3}}{\sqrt{3}} = 2$.
7. $\dfrac{\sqrt{2}}{\sqrt[3]{2}} = \dfrac{2^{1/2}}{2^{1/3}} = 2^{1/2-1/3} = 2^{1/6} = \sqrt[6]{2}$.

EXERCISE 19.6.2

Write as a single radical in simplest form (or without a radical or fractional exponent if possible):

1. $\sqrt{5} \cdot \sqrt{6}$.
2. $\sqrt{10} \cdot \sqrt{6}$.
3. $\sqrt{6} \cdot \sqrt{12}$.
4. $\sqrt{10} \cdot \sqrt{15}$.
5. $\sqrt{3a} \cdot \sqrt{15a}$.
6. $\sqrt{6a^3} \cdot \sqrt{6a}$.
7. $\sqrt[3]{3} \cdot \sqrt[3]{18}$.
8. $\sqrt[3]{25a} \cdot \sqrt[3]{5a^2}$.
9. $\sqrt[3]{9x^4} \cdot \sqrt[3]{-3x^2}$.
10. $\sqrt[3]{-2x^4} \cdot \sqrt[3]{12x^5}$.

11. $\sqrt{3}\cdot\sqrt[3]{3}$.

12. $\sqrt[3]{2t^2}\cdot\sqrt{8t}$.

13. $\sqrt[3]{6a^7}\cdot\sqrt[4]{16a^5}$.

14. $\dfrac{\sqrt{32}}{\sqrt{2}}$.

15. $\dfrac{\sqrt{18}}{\sqrt{10}}$.

16. $\dfrac{\sqrt{180}}{\sqrt{10}}$.

17. $\dfrac{\sqrt{162a}}{\sqrt{6a}}$.

18. $\dfrac{\sqrt{50a}}{\sqrt{2a^3}}$.

19. $\dfrac{2x\cdot\sqrt{54x}}{\sqrt{30x^3}}$.

20. $\dfrac{6y\cdot\sqrt{2x^3y}}{5x\cdot\sqrt{8xy^3}}$.

21. $\dfrac{2\cdot\sqrt[3]{5}}{5\cdot\sqrt[3]{2}}$.

22. $\dfrac{2\cdot\sqrt[3]{25}}{5\cdot\sqrt[3]{4}}$.

23. $\dfrac{x\cdot\sqrt[3]{5x}}{3\cdot\sqrt[3]{4x^2}}$.

24. $\dfrac{\sqrt[3]{x^2}}{\sqrt{x^3}}$.

25. $\dfrac{\sqrt{2x}\cdot\sqrt{18x^3}}{\sqrt{10x}}$.

26. $\dfrac{2s\cdot\sqrt{27s}\cdot\sqrt{54s^3}}{\sqrt{50s}\cdot\sqrt{125}}$.

Multiply:

27. $(3-\sqrt{2})(3+\sqrt{2})$.
28. $(\sqrt{3}-2)(\sqrt{3}+2)$.
29. $(x-\sqrt{2})(x+\sqrt{2})$.
30. $(\sqrt{3}-\sqrt{2})(\sqrt{3}+\sqrt{2})$.
31. $(2\sqrt{5}-3\sqrt{2})(2\sqrt{5}+3\sqrt{2})$.
32. $(x\sqrt{y}-y\sqrt{x})(x\sqrt{y}+y\sqrt{x})$.
33. $[(x-2)+\sqrt{3}][(x-2)-\sqrt{3}]$.
34. $[x-(4-\sqrt{2})][x-(4+\sqrt{2})]$.
35. $\left(x-\dfrac{2+\sqrt{3}}{2}\right)\left(x-\dfrac{2-\sqrt{3}}{2}\right)$.
36. $(x-\sqrt{3})^2$.
37. $(2x+\sqrt{5})^2$.
38. $(x+\sqrt{2}-\sqrt{3})^2$.

19.7 Rationalizing Denominators

You may have been a little surprised when you did Problems 23 and 24 of Exercise 19.3.1 to discover that you got the same answer for each. In doing Problem 23 you eventually had the problem of dividing 3.146 by 1.732, whereas for 24 your final problem was to divide 5.449 by 3. There is no doubt which of these is easier. For this reason (and sometimes other reasons) when we have an expression with a radical in the denominator it is desirable to change it to an equal expression without a radical in the denominator. This is called "rationalizing the denominator." The principle involved is the one that we learned long ago, $a/b = ax/bx$, that is, we may multiply numerator and denominator by the same (non-zero) number. The number that we choose is one which makes it possible to write the denominator without a radical.

EXAMPLES 19.7.1

1. $\dfrac{3}{\sqrt{2}} = \dfrac{3 \cdot \sqrt{2}}{\sqrt{2} \cdot \sqrt{2}} = \dfrac{3\sqrt{2}}{2}.$

2. $\dfrac{2\sqrt{3}}{\sqrt{2}} = \dfrac{2\sqrt{3} \cdot \sqrt{2}}{\sqrt{2} \cdot \sqrt{2}} = \dfrac{2\sqrt{6}}{2} = \sqrt{6}.$

3. $\dfrac{\sqrt[3]{3}}{\sqrt[3]{2}} = \dfrac{\sqrt[3]{3} \cdot \sqrt[3]{2^2}}{\sqrt[3]{2} \cdot \sqrt[3]{2^2}} = \dfrac{\sqrt[3]{12}}{2}.$

4. $\dfrac{3}{\sqrt[3]{x^2}} = \dfrac{3 \cdot \sqrt[3]{x}}{\sqrt[3]{x^2} \cdot \sqrt[3]{x}} = \dfrac{3 \cdot \sqrt[3]{x}}{x}.$

5. $\dfrac{4}{1 + \sqrt{2}} = \dfrac{4(1 - \sqrt{2})}{(1 + \sqrt{2})(1 - \sqrt{2})} = \dfrac{4(1 - \sqrt{2})}{1 - 2}$
$= 4(\sqrt{2} - 1).$

6. $\dfrac{4}{\sqrt{5} - 1} = \dfrac{4(\sqrt{5} + 1)}{(\sqrt{5} - 1)(\sqrt{5} + 1)} = \dfrac{4(\sqrt{5} + 1)}{5 - 1}$
$= \sqrt{5} + 1.$

7. $\dfrac{\sqrt{a} + \sqrt{b}}{\sqrt{a} - \sqrt{b}} = \dfrac{(\sqrt{a} + \sqrt{b})(\sqrt{a} + \sqrt{b})}{(\sqrt{a} - \sqrt{b})(\sqrt{a} + \sqrt{b})}$
$= \dfrac{(\sqrt{a} + \sqrt{b})^2}{a - b}.$

EXERCISE 19.7.2

Rationalize the denominators, and, using tables, find the decimal approximations for Problems 1 through 18:

1. $\dfrac{2}{\sqrt{3}}.$
2. $\dfrac{3 \cdot \sqrt{2}}{\sqrt{5}}.$
3. $\dfrac{1}{\sqrt[3]{3}}.$
4. $\dfrac{\sqrt[3]{2}}{\sqrt[3]{9}}.$

5. $\dfrac{2}{\sqrt[3]{2}}$. 6. $\dfrac{3}{\sqrt[3]{16}}$. 7. $\sqrt{\frac{4}{3}}$. 8. $\sqrt{\frac{2}{5}}$.

9. $\dfrac{2}{3-\sqrt{2}}$. 10. $\dfrac{6}{\sqrt{7}-3}$. 11. $\dfrac{\sqrt{2}}{\sqrt{2}-1}$. 12. $\dfrac{4\sqrt{12}}{\sqrt{6}-2}$.

13. $\dfrac{4}{\sqrt{6}+\sqrt{2}}$. 14. $\dfrac{3+\sqrt{3}}{2-\sqrt{3}}$.

15. $\dfrac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}$. 16. $\dfrac{4\cdot\sqrt{2}+3\cdot\sqrt{3}}{4\cdot\sqrt{2}-3\cdot\sqrt{3}}$.

17. $\dfrac{\sqrt{2}+3\cdot\sqrt{3}}{3\cdot\sqrt{2}+2\cdot\sqrt{3}}$. 18. $\dfrac{3\cdot\sqrt{2}+\sqrt{3}}{2\cdot\sqrt{5}+\sqrt{7}}$.

Rationalize the denominators:

19. $\dfrac{1}{\sqrt{2a}}$. 20. $\dfrac{2a}{\sqrt{3a}}$. 21. $\dfrac{6x^2}{\sqrt[3]{4x}}$.

22. $\sqrt{\dfrac{2c}{3x}}$. 23. $\sqrt{\dfrac{5t}{6s}}$. 24. $\dfrac{\sqrt{a+b}}{\sqrt{a-b}}$.

25. $\dfrac{\sqrt{x-y}}{\sqrt{x}}$. 26. $\sqrt{\dfrac{2x}{x-y}}$. 27. $\sqrt[3]{\dfrac{x}{y^2}}$.

28. $\sqrt{\dfrac{5a}{8b^3}}$. 29. $\dfrac{\sqrt{x}+\sqrt{2y}}{\sqrt{x}-\sqrt{2y}}$. 30. $\sqrt{\dfrac{b^2}{4a^2}-\dfrac{c}{a}}$.

31. Find the hypotenuse of a right triangle, correct to 3 decimal places, if the legs are 3 and 5, respectively.

32. The hypotenuse of a right triangle is 8 feet, and one leg is 5 feet. How long is the other leg?

33. A rectangular grass plot is 100 feet long and 50 feet wide. If we wish to go from one corner to the diagonally opposite corner, how much distance do we save by not obeying the "Keep Off the Grass" sign?

34. A ladder 10 feet long just reaches a window when the bottom of the ladder is 4 feet from the wall. How high above the ground is the window?

20

The Complex Number System

20.1 Pure Imaginary Numbers

We have seen that the question, What is the solution of the equation $x^2 = 2$? made it necessary, if we wanted an answer, to introduce a new kind of number, the irrational numbers. We then noted that these numbers, along with the rational numbers, form a system of numbers whose distinguishing property is that the square of each of them is either positive or zero. To this system of numbers we gave the name real numbers.

If we consider the equation $x^2 + 4 = 0$ or the equivalent equation $x^2 = -4$, we have a choice. There is no real number whose square is negative 4; we can either agree to say that no value of x can be found to make the equation a true statement or we can boldly invent a new type of number. By now we should have the temerity to take the second choice and invent a new kind of number, a number whose square is negative.

20.1.1 DEFINITION: A *pure imaginary* number is a number whose square is negative.*

20.1.2 DEFINITION: The number whose square is -1 will be denoted by i, so that $i^2 = -1$, and, if a is any other number, then $(ai)^2 = a^2i^2 = -a^2$. We call i the imaginary unit.†

---

* Again we meet an unfortunate terminology. There is nothing more "imaginary" about an imaginary number than there is about a real number. We invented both of them. "Imaginary number" is another of those technical terms which have nothing to do with the ordinary meaning of the words. Any electrical engineer will tell you how "real" an imaginary number is to him. He would have great difficulty in solving some of his problems without the use of these numbers.

† Physicists and electrical engineers use the letter j instead of i for the imaginary unit. This is because they use the letter i as their symbol for electrical current.

We may now say that the solutions of the equation $x^2 = -4$ are $2i$ and $-2i$ since $(2i)^2 = 2^2i^2 = -4$ and $(-2i)^2 = (-2)^2i^2 = -4$. Every pure imaginary number can be written as the product of a real number (not zero) and i.

Examples 20.1.3

1. The solutions of $x^2 = -9$ are $3i$ and $-3i$.
2. The solutions of $x^2 = -5$ are $i \cdot \sqrt{5}$ and $-i \cdot \sqrt{5}$.*
3. Since $i^2 = -1$, then $i^3 = -i$ and $i^4 = -i^2 = 1$.

20.2 Complex Numbers

The system of numbers made up of the real numbers and the pure imaginary numbers is not closed under the operation of addition. For, if we add a real number, not zero, and a pure imaginary number, such as $3 + 5i$, we obtain a number which is neither real nor pure imaginary.

20.2.1 Definition: A number of the form $a + bi$, where a and b are real numbers, is called a *complex number*.†

If we now consider the system of numbers of the form $a + bi$, we see that it includes all our previous kinds of numbers. To distinguish among these numbers we use the following:

1. If b is zero, then we say we have a real number (a number whose square is positive or zero). Real numbers themselves are positive, negative, or zero. $3, 3\sqrt{7}, -5, -3\sqrt{2}$ are real numbers.

---

* When the real factor of a pure imaginary number involves a radical, it is best to write the i before the radical. There is then less danger of the i sliding under the radical.

† We have used the word "complex" also in "complex fraction" (12.2). These two phrases, complex number and complex fraction, refer to two quite different things. Of course, a complex fraction may involve complex numbers.

$$\frac{\frac{1}{2} + \frac{1}{3}}{\frac{1}{2} - \frac{1}{3}}$$

is a complex fraction, and

$$\tfrac{1}{2} + \tfrac{1}{3}i$$

is a complex number.

$$\frac{\frac{1}{2} + \frac{1}{3}i}{\frac{1}{2} - \frac{1}{3}i}$$

is a complex fraction involving complex numbers.

2. If a is zero, then we have a pure imaginary number (provided that b is not also zero). $3i$, $2i \cdot \sqrt{7}$, $-5i$, $-3i \cdot \sqrt{2}$ are pure imaginary numbers.

3. If neither a nor b is zero then we have a *complex imaginary number*. $3 + 2i$, $\sqrt{2} - 5i$, $4 - i \cdot \sqrt{5}$, $\sqrt{3} + i \cdot \sqrt{7}$ are complex imaginary numbers; a is called the real part, and b is called the pure imaginary part of $a + bi$.

The algebra of complex numbers is precisely the same as that of real numbers, except that, if we obtain i^2, we replace it by -1. A complex number will be in simplest form when it is in the form $a + bi$, where a and b are real numbers.

EXAMPLES 20.2.2

1. $(3 + 2i) + (4 - 5i) = 7 - 3i$.
2. $(3 + 2i) - (4 - 5i) = -1 + 7i$.
3. $(3 + 2i)(3 - 2i) = 9 - 4i^2 = 9 + 4 = 13$.
4. $(1 + i)^2 = 1 + 2i + i^2 = 1 + 2i - 1 = 2i$.
5. $(1 + i)^4 = 4i^2 = -4$.
6. $(2 + 3i)(5 - 4i) = 10 + 7i - 12i^2 = 10 + 7i + 12 = 22 + 7i$.
7. $(2 + 3i)^2 = 4 + 12i + 9i^2 = -5 + 12i$.

We note that the square of a complex imaginary number is neither positive, negative, nor zero. What kind of number is it?[36]

20.2.3 DEFINITION: Two complex numbers $a + bi$ and $c + di$ are equal, if and only if $a = c$ and $b = d$.

20.2.4 DEFINITION: The complex numbers $a + bi$ and $a - bi$ are said to be *conjugate* complex numbers.

EXAMPLES 20.2.5

1. If $x + iy = 3 - 2i$, then $x = 3$ and $y = -2$.
2. If $2x + 3y + 5xi + 2yi = 4 + 3i$, then $2x + 3y = 4$ and $5x + 2y = 3$.
3. If $a + bi = 0 = 0 + 0 \cdot i$, then $a = 0$ and $b = 0$.
4. $5 + 2i$ and $5 - 2i$ are conjugate complex numbers.
5. $i \cdot \sqrt{3}$ and $-i \cdot \sqrt{3}$ are conjugate complex numbers.
6. 4 and 4 are conjugate complex numbers.

EXERCISE 20.2.6

Write in the form bi, where b is a real number:

1. $\sqrt{-4}$. 2. $\sqrt{-25}$. 3. $\sqrt{-8}$.
4. $\sqrt{-27}$. 5. $\sqrt{-50}$. 6. $3 \cdot \sqrt{-18}$.

Write in simplest form:

7. $3 + \sqrt{-9}$. 8. $4 - \sqrt{-3}$.
9. $5 + 2 \cdot \sqrt{-32}$. 10. $-2(-3 + 2 \cdot \sqrt{-5})$.
11. i^5, i^6, i^7, i^8. 12. $i^{17}, i^{18}, i^{19}, i^{20}$.
13. $i^{57}, i^{58}, i^{59}, i^{60}, i^{127}$. 14. $(2 + i) + (2 - i)$.
15. $5 + 2i + 3 - 4i - 7$. 16. $(3 - 2i) - (3 + 2i)$.

17. $3 + \sqrt{-2} + \sqrt{-9}$.
18. $(2 - \sqrt{-8}) + (-4 + \sqrt{-27})$.
19. $\sqrt{18} + \sqrt{-25} - \sqrt{25} - \sqrt{-18}$.

20. $(2 + i)(2 - i)$. 21. $(6 + 4i)(6 - 4i)$.
22. $(3 + 2i)^2$. 23. $(2 - 3i)^2$.
24. $(-2 + 3i)^2$. 25. $(3 + 2i)(2 - 3i)$.
26. $(-5 + i)(4 - i)$. 27. $i(3 + 2i)$.
28. $i^3(4 - 2i)$. 29. $i^5 \cdot i^6(2 + 2i)$.
30. $(-1 + i\sqrt{3})^2$. 31. $(-1 + i\sqrt{3})^3$.
32. $(-1 - i\sqrt{3})^3$.

What are all the solutions of the equation $x^3 = 8$? [37]

33. Show that $2 + 2i$, $-2 + 2i$, $-2 - 2i$, $2 - 2i$ are each roots of $x^4 + 64 = 0$.

34. Show that $\sqrt{3} + i$, $2i$, $-\sqrt{3} + i$, $-\sqrt{3} - i$, $-2i$, and $\sqrt{3} - i$ are each roots of $x^6 + 64 = 0$.

Find the real values of x and y satisfying each of the following:

35. $3x + 2iy = 2 + 3i$.
36. $(2x + 4y) + (3x - 2y)i = 2 + i$.
37. $x(1 + 3i) + y(3 - i) = 5(3 + i)$.
38. $x(2 + 3i) - y(7 - 2i) = 2 - 3i$.

39. Show that the sum of a complex number and its conjugate is a real number.

40. Show that the difference of a complex imaginary number and its conjugate is a pure imaginary number.

41. Show that the product of a complex number, not zero, and its conjugate is a positive real number.

20.3 Division of Complex Numbers

We extend the definition of division to complex numbers by

20.3.1 DEFINITION: The quotient of two complex numbers, $(a + bi) \div (c + di)$, $c + di \neq 0$, is a complex number $x + yi$ such that $a + bi = (c + di)(x + yi)$.

We shall now show that the complex numbers are closed under the operation of division. This means, of course, that there always is a complex number $x + yi$ such that $a + bi = (c + di) \cdot (x + yi)$ for any values of a, b, c, d $(c + di \neq 0)$.

If we take

$$x = \frac{ac + bd}{c^2 + d^2} \quad \text{and} \quad y = \frac{bc - ad}{c^2 + d^2}\ *$$

we shall have

$$\begin{aligned}
&(c + di)\left(\frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2} \cdot i\right) \\
&= c\left(\frac{ac + bd}{c^2 + d^2}\right) + di^2\left(\frac{bc - ad}{c^2 + d^2}\right) + di\left(\frac{ac + bd}{c^2 + d^2}\right) + ci\left(\frac{bc - ad}{c^2 + d^2}\right) \\
&= \frac{[c(ac + bd) - d(bc - ad)] + [d(ac + bd) + c(bc - ad)]i}{c^2 + d^2} \\
&= \frac{(ac^2 + bcd - bcd + ad^2) + (acd + bd^2 + bc^2 - acd)i}{c^2 + d^2} \\
&= \frac{a(c^2 + d^2) + b(c^2 + d^2)i}{c^2 + d^2} = a + bi,
\end{aligned}$$

as we wished to show.

In practice we obtain the quotient $(a + bi) \div (c + di)$ by a method similar to that used in rationalizing denominators (19.7). We multiply numerator and denominator by the conjugate of the denominator.

---

* We obtain these values of x and y by multiplying out the right side of $a + bi = (c + di)(x + yi)$ and then using 20.2.3.

EXAMPLES 20.3.2

1. $\dfrac{2+i}{1-i} = \dfrac{(2+i)(1+i)}{(1-i)(1+i)} = \dfrac{2+3i-1}{1+1} = \dfrac{1+3i}{2} = \dfrac{1}{2} + \dfrac{3}{2}i.$

2. $\dfrac{1+i}{1-i} = \dfrac{(1+i)^2}{(1-i)(1+i)} = \dfrac{2i}{2} = i.$

3. $\dfrac{2-3i}{i} = \dfrac{(2-3i)(-i)}{i(-i)} = -3 - 2i.$

EXERCISE 20.3.3

Write in simplest form:

1. $\dfrac{2-3i}{3+2i}.$ 2. $\dfrac{1-i}{1+i}.$ 3. $\dfrac{1}{i}.$ 4. $\dfrac{3+i}{2i-1}.$

5. $\dfrac{4}{2-i}.$ 6. $\dfrac{3+2i}{3-2i}.$ 7. $\dfrac{5-2i}{3i}.$ 8. $\dfrac{3+i}{(2+i)^2}.$

Find real values of x and y satisfying

9. $x + yi = \dfrac{5+10i}{1-2i}.$ 10. $1 - 2i = \dfrac{2+3i}{x+yi}.$

Write in the simplest form:

11. $\dfrac{\frac{1}{2}+\frac{1}{3}i}{\frac{1}{2}-\frac{1}{3}i}.$ (Hint: First change the complex fraction to a simple fraction.)

12. $\dfrac{i}{\frac{1}{2}+\frac{2}{5}i}.$ 13. $\dfrac{\frac{i}{2}}{\frac{3}{i}}.$ 14. $\dfrac{\frac{1}{2i}-\frac{1}{3}}{\frac{1}{2i}+\frac{1}{3}}.$ 15. $\dfrac{\frac{2}{3i}+\frac{3i}{2}}{\frac{2}{3i}-\frac{3i}{2}}.$

16. $\dfrac{\frac{1}{a}+\frac{1}{bi}}{\frac{1}{ai}+\frac{1}{b}}.$ (a and b real.)

Appendix

This appendix consists of answers to the questions which appear at irregular intervals in the text. In some cases comments in addition to the answers are given. It is hoped that these comments will inspire some students to investigate the matters further.

The number before the parentheses corresponds to the number of the question as it appears in the text; the number within the parentheses is the number of the part of the text immediately preceding the question or containing it.

1(1) In clock arithmetic $8 + 8 = 4$. Our question is equivalent to asking, "At what time is it 8 hours after 8 o'clock?" We obtain our result by adding $8 + 8$, as in ordinary arithmetic, and subtracting 12 from this sum. To find the clock arithmetic sum of two numbers we add them as in ordinary arithmetic, and, if this sum is less than 12, then it is also the "clock sum"; if this sum is more than 12, we subtract 12 from it to obtain the "clock sum." $9 + 2 = 11$ by ordinary and also by clock arithmetic. $7 + 6 = 13$ by ordinary arithmetic, but $7 + 6 = 13 - 12 = 1$ by clock arithmetic.

We may invent other clock arithmetics in which the "clock" has 24 hours or 5 hours or 17 hours or any other natural number of hours on it.

You will find this arithmetic discussed more fully in Richardson's *Fundamentals of Mathematics*, page 161.*

2(2.2.1) The set of natural numbers is not closed under division since, for instance, $5 \div 2$ is not a natural number.

---

* For a still more complete discussion look up "congruences" in Trygve Nagell, *Introduction to Number Theory*, John Wiley & Sons, 1951, or Harry N. Wright, *First Course in the Theory of Numbers*, John Wiley & Sons, 1939. This part of Theory of Numbers is due primarily to Carl Friedrich Gauss, known as "The Prince of Mathematicians." You can read his interesting life story in E. T. Bell, *Men of Mathematics*, Simon and Schuster, N. Y., 1937.

3(2.2.1) The set of even natural numbers is closed under addition. The sum of any two even natural numbers is an even natural number.

4(2.2.1) The set of odd natural numbers is not closed under addition. The sum of two odd numbers is never an odd number.

5(2.2.2) The operation of division is not commutative. For an operation to be commutative we must get the same result when we apply it to two numbers taken in one order as we get when we apply it to these same numbers in the reverse order. Five divided by 2 is not the same as 2 divided by 5.

6(2.5.2) By analogy with Postulate 2.2.2 we call this the commutative law for multiplication of natural numbers. Any law which states that an operation on two numbers gives the same result regardless of the order of the numbers is a commutative law. The operation of writing two numbers next to each other is not commutative; 23 is not the same as 32. In electricity a commutator is a gadget for reversing the direction of an electric current.

7(2.5.3) By analogy with Postulate 2.2.3 we call this the associative law for multiplication of natural numbers.

8(5.1.5) Problem 15 states: $(x + y)^2 = x^2 + 2xy + y^2$, where x and y are (natural) numbers. If we substitute 2 for x, and 3 for y, we have

$$\begin{aligned}(2 + 3)^2 &= 2^2 + 2\cdot 2\cdot 3 + 3^2\\ &= 4 + 12 + 9 = 25\end{aligned}$$

and also

$$(2 + 3)^2 = 5^2 = 25.$$

An equality such as that in Problem 15 is called an identity; it becomes a true statement no matter what numbers are substituted for x and y. Each of Problems 10 through 24 of Exercise 5.1.5 is an identity. You might try substituting numbers for the letters in some of them.

9(7.1.7) Equations 44. $(x - 5)(x - 11) = 0$ and 48. $(x - 5)(x - 11)(x - 13) = 0$ are not equivalent. Although every solution of Equation 44 (i.e., 5 and 11) is a solution of Equation 48, there is a solution of Equation 48 (i.e., 13) which is not a solution of Equation 44. For two equations to be equivalent, it must be true that every solution of the first is a solution of the second *and* every solution of the second is a solution of the first.

10(8.7.4) To move from -3 to $+5$ we must move 8 units to the right.

11(8.7.4) Since subtracting a number b from a number a is the same as adding the negative of b to a, each time we have a problem which we say leads to subtraction, we could instead say that it leads to addition of the negative of the subtrahend. If we have a pole 10 feet long and cut off a 3-foot length, instead of saying we obtain the new length by subtracting 3 from 10, we could say that we add -3. Each subtraction problem may be reworded to make it an addition problem, so that we need never use the operation of subtraction.

12(10.1.4) Since $63 \div 9$ is an integer, 7, or $9 \cdot 7 = 63$.

13(10.1.4) Since $10 \div 10$ is an integer, 1, or $10 \cdot 1 = 10$.

14(10.1.4) Since $(-48) \div 6$ is an integer, -8, or $6 \cdot (-8) = -48$.

15(10.1.4) Since $(-33) \div (-3)$ is an integer, 11, or $(-3) \cdot 11 = -33$.

16(10.1.4) Since $(-33) \div (-11)$ is an integer, 3, or $(-11) \cdot 3 = -33$.

17(10.1.4) Other divisors or factors of 30 are: 1, 2, 3, 10, 15, 30, -1, -2, -3, -5, -6, -10, -15, -30.

18(11.1.3) Substituting 1.2 for x in $(x - 1)/(x^2 - 1)$ we have

$$\frac{1.2 - 1}{1.44 - 1} = \frac{.2}{.44} = \frac{20}{44} = \frac{5}{11};$$

for $x = 1.1$ we have

$$\frac{1.1 - 1}{1.21 - 1} = \frac{.1}{.21} = \frac{10}{21}.$$

The other values for x give

$$\tfrac{100}{201}, \tfrac{1000}{2001}, \tfrac{100}{199}, \tfrac{1000}{1999}, \text{ respectively.}$$

It seems that as x gets close to 1 the fraction

$$(x - 1)/(x^2 - 1) \text{ gets close to } \tfrac{1}{2}.$$

Mathematicians say that the limit of $(x - 1)/(x^2 - 1)$ as x approaches 1 is $\frac{1}{2}$. This means just what we said, no more and no less: when x is close to 1 but not equal to 1, the fraction

$(x - 1)/(x^2 - 1)$ is close to $\frac{1}{2}$. It does not mean that $(x - 1)/(x^2 - 1) = \frac{1}{2}$ when $x = 1$. That would be nonsense.

19(11.8.2) Some fractions closer to $\frac{1}{2}$ than 499/1000 are

$$4999/10{,}000;\ 49{,}999/100{,}000;\ 5001/10{,}000;\ 50{,}001/100{,}000.$$

20(12.4.1) 37/59 will appear in the 59th row and will be the 37th number to the right of 0/59; −427/999 will appear in the 999th row and will be the 427th number to the left of 0/999.

Of course, it would not be possible to actually write down all the rational numbers in this way. But, in the sense that each rational number has a definite position in this display, we may say that all rational numbers are contained in the display, just as all natural numbers are contained in 1, 2, 3, 4, $\cdots$.

21(13.2) The square of the sum of two numbers equals the square of the first number plus twice the product of the two numbers plus the square of the second number.

The square of the difference of two numbers equals the square of the first number minus twice the product of the two numbers plus the square of the second number.

22(13.3.1) At the end of 1 second the stone is $120 - 16 = 104$ feet high. At the end of 2 seconds it is $120 \cdot 2 - 16 \cdot 4 = 176$ feet high; 3 seconds, 216 feet high; 4 seconds, 224 feet high; $\frac{3}{2}$ second, 144 feet high; $\frac{2}{3}$ second, $\frac{656}{9}$ feet high.

23(13.3.1) There is no limit to the number of such questions which may be asked. We might, for instance, ask how high the stone is after 9 seconds; we obtain the perhaps puzzling answer −216 feet. To interpret this we may think of throwing the stone up from the roof of a tall building. After 9 seconds the stone has reached its maximum height, fallen, and is, at the end of 9 seconds, 216 feet lower than the roof of the building.

24(13.3.1) s and t may take on an unlimited set of values; for each t there is a corresponding s.

25(14.1.1) These can be verified by multiplying the factors on the right.

26(15.1) We can see this, if we list all the possible matches. If the players are Albert, Bill, Carl, and David, we shall have Albert vs. Bill, Albert vs. Carl, Albert vs. David, Bill vs. Carl, Bill vs. David, Carl vs. David.

Also we can see this by first thinking of the number of ways in which we can pick the first player and, after he is picked, the number of ways in which we can pick the second player. Since there are four persons involved, we can pick the first player as any one of the four or in four ways. After the first is picked, there are three to choose from for his opponent so that the opponent can be picked in three ways. There will then be $4 \cdot 3 = 12$ ways of arranging the matches, if it makes any difference who is picked first. But, of course, this is of no significance; a match between Albert and Bill is the same as a match between Bill and Albert. We have twice as many as we should have, the correct number being 6.

27(15.1) We can show that $n(n-1)/2$ gives the correct number of matches for n players by the method used in the special case above, where $n = 4$.

If there are n players, we can choose the first of a match in n ways. There will then be $n - 1$ players among which to choose the opponent. There are $n(n - 1)$ ways of making a match, if the order of choice is significant. But again this is of no importance, and the number of matches is $n(n - 1)/2$.

An example in which the order of choice is significant is the answer to the question, How many 2-digit numbers can be made using the digits 1, 2, 3, 4 only, repetitions not being permitted? As before, we may choose the first in four ways and then the second in 3 ways, or the two in $4 \cdot 3 = 12$ ways. But here the order of choice is significant. If we choose the 2 first and then the 3, we get 23; if we reverse the order of choice, we get 32. There are 12 different 2-digit numbers that we can form from the digits 1, 2, 3, 4. They are 12, 13, 14, 21, 31, 41, 23, 24, 32, 42, 34, 43.

To find out more about the topic we have been discussing in these last two answers, look up permutations and combinations in any college algebra book or see Richardson: *Fundamentals of Mathematics*, page 494.

28(15.1) n must be a positive integer not less than 2. Obviously we cannot have a match involving just one person; at least it would be difficult.

29(18.2.4) Equations (*b*), (*c*), (*d*), (*e*), and (*f*) are each equivalent to each other. The only solutions of each are 1 and 2.

30(18.2.4) We subtracted 1 from each side.

31(18.2.4) Because 4 is not in the range of the functions on either side of the equals sign. These functions become meaningless for $x = 4$.

32(18.2.5) Equations (b) through (g) are equivalent to each other; 2 and 4 are roots of each of these equations and the only roots. Equation (a) is not equivalent to any of the others because, although 4 is a root of it, 2 is not a root of it.

33(18.2.5) All these equations are equivalent to each other; 2 and 3 are roots of each and the only roots of each.

34(18.3.1) We used an incorrect operation in going from (a) to (b). Of course, we may divide the numerator and denominator of a fraction by the same number. But we have not done this here; we have divided one term of the numerator and one term of the denominator by 2.

35(19.3) The only numbers which are exact square roots are those opposite perfect squares in the first column; the exact positive square root of 9 is 3.000. Similarly the only numbers which are exact cube roots are those opposite perfect cubes; the exact cube root of 8 is 2.000.

All other numbers are 3-decimal-place rational approximations to numbers which themselves are irrational. You might try squaring 3.317 to see how close its square is to 11 and cubing 5.313 to see how close its cube is to 150.

36(20.2.2) It is an imaginary number, either pure imaginary or complex imaginary. To see this we let $a + bi$ be any complex imaginary number; this means that $a \neq 0$ and $b \neq 0$. Then

$$(a + bi)^2 = a^2 + 2abi + b^2i^2 = (a^2 - b^2) + 2abi.$$

Since neither a nor b is zero, $2ab \neq 0$, and hence the imaginary part is not zero. The real part, $a^2 - b^2$, will be zero if and only if $a = b$ or $a = -b$. We have shown that the square of a complex imaginary number is a complex imaginary number unless $a = b$ or $a = -b$, in which case the square is a pure imaginary number.

37(20.2.6) The solutions of $x^3 = 8$ are 2, $-1 + i\sqrt{3}$, and $-1 - i\sqrt{3}$. We verify this by noting that $2^3 = 8$ and by Problems 31 and 32. We have then three cube roots of 8. In Problem 33, we have four fourth roots of -64; in Problem 34 we have six sixth roots of -64. We might guess that each number has n nth roots. This would be a good guess.

Answers

Answers are given to all odd-numbered problems except for those in which the answer can be easily checked and also where for obvious reasons an answer should not be given. Answers to even-numbered problems are available to instructors from the publisher.

Exercise 2.3.2

11. $(a + b) + c = c + (a + b)$.
13. $[(a + b) + c] + d = a + [b + (c + d)]$.

Exercise 2.4.3

17. $25 \cdot 7$.
19. $11(3s)$.
21. $(r + s)(x + y)$.

Exercise 2.5.4

13. 35.
15. 63.
17. 29.
19. 112.
21. 28.
23. 28.
35. 420.
37. 137.

Exercise 3.1.2

1. $n + 7 = 15$.
3. $n + 3n = 56$.
5. $a + 2a = 42$.
7. $x + 2x = 42$.
9. $2e + e + (2e + 4) = 59$.
11. $n + 5n + 10n = 80$.
13. $n(n + 1) = 132$.
15. $30 + 2a + a = 180$.
17. $w + w + 2w + 2w = 66$ or $2(w + 2w) = 66$.
19. $w(4w) + 140 = (w + 2)(4w + 5)$.
21. $2(60 + w) = 5w$.
23. $P + (P + 60) = 454$.
25. $10t = 3(t + 6)$.

Exercise 4.1.3

1. 171.
3. 5035.
5. 12,882.
7. $\overline{xy} + \overline{xz} + \overline{xw}$.
9. $(10xy) + (15xz) + (20xw)$.
11. $(xp) + (yp) + (xq) + (yq) + (xr) + (yr)$.

13. $(xp) + (yp) + (zp) + (xq) + (yq) + (zq) + (xr) + (yr) + (zr) + (xs) + (ys) + (zs)$.
15. $(8am) + (12bm) + (10an) + (15bn)$.
17. $(sa) + (sbc) + (sbd)$.
19. $(6ts) + (4tw) + (4tv)$.
21. $(xx) + (yy) + (zz) + (xy) + (xy) + (yz) + (yz) + (xz) + (xz)$.

23. $(4aa) + (6ab) + (6ab) + (9bb)$.
25. $(9pp) + (6pq) + (6qp) + (4qq)$.
27. $7(5 + 9)$.
29. $7(a + 1)$.
31. $x(y + z)$.
33. $2x(y + z + w)$.
35. $(y + z)(x + p)$.
37. $[y + (2z)][(3x) + (5w)]$.

Exercise 4.2.3

1. 19.
3. 45.
5. 37.
7. 5370.
9. $ab + ac + d + e + f + gf$.
11. $3 + 4 \cdot 5 + 6$.

13. $3 + 4 \cdot 5 + 6 \cdot 7(8 + 9) + 2 \cdot 3 + 4$.
15. $sx(y + z) + s\{(x + y)(y + z) + x + yz\}$.

17. $s[a + b(c + d)]$.
19. $2t[3s + 2(w + v)]$.
21. $(x + y + z)(x + y + z)$.
23. $[2a + 3b][2a + 3b]$.
25. $[3p + 2q][3p + 2q]$.

Exercise 5.1.5

25. 18.
27. 36.
29. 25.
31. 98.
33. 22.
35. 302.
37. 3602.
39. 676.
41. 436.
43. $x^2 + xy$.
45. $x^2y + xy^2$.
47. $5x^3 + 3x^2y$.

49. $x^3y + x^2y^2$.
51. $x^3y + xy^3$.
53. $x^4y^2 + x^2y^4$.
55. $x^3 + 2x^2y + xy^2$.
57. $x^4y^2 + 2x^3y^3 + x^2y^4$.
59. $x^5y^2 + 3x^4y^3 + 3x^3y^4 + x^2y^5$.

61. $x^3y^2 + x^2y^3$.
63. $x^5 + x^4y$.
65. $x^5y^4 + x^4y^5$.
67. $x^8 + x^6y^2$.
69. $x^9y^6 + x^6y^9$.
71. $y^2(x + 2)$.
73. $y^2(x^2 + 3)$.
75. $x^2y(1 + y)$.

77. $x^2y^2(3 + 2) = 5x^2y^2$.
79. $2^5x^2y^3(y^3 + 2x^2)$.
81. $8x^2y(2x + y^2)$.
83. $x(y + 3 + 4y^2)$.
85. $(x + y)(x + y) = (x + y)^2$.
87. $(x + y)^2$.
89. $(x + 2y)(x + y)$.
91. $xy(x + y)^2(x + y) = xy(x + y)^3$.
93. 65.
95. 180.
97. 156.
99. 636.

Exercise 5.2.5

1. x^7.
3. a^8.
5. a^{10}.
7. x^{15}.
9. r^6.
11. $a^{10}b^5$.
13. $3^4x^2 = 81x^2$.
15. x^{2m}.
17. $(ab^2)^t$.
19. x^{3m}.
21. $(a + b)^4$.
23. $a^5x^5 = (ax)^5$.

25. r^7s^8.
27. $a^8b^8c^8 = (abc)^8$.
29. $p^4q^4r^3$.
31. $x^{12}q^6$.
33. $x^5 + x^3y + x^2y^3$.
35. $p^7q^7r^7 + 2p^6q^6r^8$.
37. $x^8y^7 + x^{10}y^5$.
39. $s^3t + s^3t^2 + s^2t^3$.

Exercise 5.2.7

1. $xy(x + y)$.
3. $ab^2c(c^2 + a^2)$.
5. $x^2y^3(3y^2 + 5x + x^2z)$.
7. $9pq^2(r^2 + 3p^2 + 9qr^3)$.
9. $3p^4q^3(5 + 7p^3qr^2)$.
11. $25b^3(ab^5c^3 + 3a^2b^2 + 5c^3)$.

13. $4r^2st(32rst + 8r^3st^2 + 1)$.
15. $a^2(a + b)^2[(a + b) + 3a] = a^2(a + b)^2[4a + b]$.

17. $(x + y)^2(1 + x + y)$.
19. $(x + y)(x + y) = (x + y)^2$.
21. $(x + y)^2$.
23. $(x + 1)^2 = 16$.
25. $n^2 + (n + 1)^2 = 25$.
27. $a^2 + 4(a + 1)^2 = 169$.
29. $c^3 + 2(c + 1)^3 = 155$.
31. $s^2 + (s + 1)^2 = 25$.
33. $r^2 + (r + 1)^2 = 29^2 = 841$.

Exercise 5.3.2

1. $13a + 12b$.
3. $4st + 3su + 5ut$.
5. $3x^2 + 5xy + 2y^2$.
7. $3p^3 + 5p^2q + 6pq^2 + q^3$.
9. $12p + 25q + 22$.
11. $3r^3s^2 + 3r^4s + 3r^4t$.

13. $18 + 11a + 13k$.
15. $a^2b + a^2c + b^2c + b^2a + c^2a + c^2b$.
17. $8p^4 + 10p^3 + 6p^2 + 8p$.
19. $5a^7b + a^6b^2 + a^5b^3 + a^4b^4 + a^3b^5 + a^2b^6$.

21. $3x^2 + 4x + 2$.
23. $3x^4 + 7x^3 + 2x^2 + 3x$.
25. $3x^4 + 8x^3 + 5x^2$.
27. $4x^5 + 6x^4 + 36x^3 + 120x^2$.
29. $2x^3 + 3x^2 + x$.

Exercise 6.1.1

1. 9.
3. 8.
5. 32.
7. 61.
9. 391.
11. 344.
13. 320.
15. 2919.
17. 2783.
19. 199,706.

Exercise 6.3.2

21. 2.
23. 1.
25. 15.
27. 38.
29. 4.
31. 1.
33. 21.
35. 12.
37. 12.
39. 32.
41. 16.
43. 16.
45. 279.
47. 133.
49. 65.
51. 65.
53. 63.
55. 63.

EXERCISE 6.4.2

1. $2n + 1 = 43.$
3. $n + (n - 3) = 33.$
5. $(n - 4) + n + [n + (n - 4)] = 40.$
7. $9 + n^2 = (n + 1)^2.$
9. $20^2 + (h - 8)^2 = h^2.$
11. $3p - 16 = p.$
13. $2[w + (w + 9)] = 70.$
15. $x + (x + 14) = 42.$
17. $8(2p) - 5p = 44.$
19. $g + (g + 4) + (g - 6) = 142.$
21. 18.
23. 22.
25. 163.
27. 22.

EXERCISE 7.1.7

17. $4 < x < 7.$
19. $0 \leqq x < 3.$
21. $1 \leqq x \leqq 15.$
23. $0 < x < 5.$
25. 17.
27. 11.
29. 5.
31. 0.
33. 16.
35. 11.
37. 10.
39. 13.
41. 59.
43. 3, 6.
45. 18, 21.
47. $x = 2, y = 4.$
49. 5, 11. (In this case 5 is said to be a *double* solution.)
51. 5, 11.
53. 9.
55. 13.
57. 10, 15.
59. 49.
61. 3.
63. 11.
65. 13.
67. 59.
69. 18, 21.
71. 5, 11, 13.
73. 43, 33.

EXERCISE 8.4.3

1. $+17.$
3. $+3.$
5. $-2.$
7. $-23.$
9. $-28.$
11. $+12.$
13. $+12.$
15. 0.
17. $-23.$
19. $-8.$
21. $-1.$
23. 0.
25. $-6.$
27. $-8.$

EXERCISE 8.5.2

13. 4.
15. 9.
17. 4.
19. 26.
21. 26.
23. 2.

EXERCISE 8.5.7

21. 0.
23. 130.
25. $-10.$
27. 82.
29. 8.
31. $-59.$
33. 88.
35. 0.
37. $-133.$
39. $-480.$
41. $-12{,}549.$

EXERCISE 8.6.4

3. 3.
5. $-9.$
7. $-23.$
9. 0.
11. 18.
13. 28.
15. 43.
17. 32.
19. 32.
21. 0.
23. 3.
25. $-27.$
27. 1.
29. $-1.$
31. $-5.$
33. $-9.$

EXERCISE 8.7.2

5. 16.
7. -28.
9. -219.
11. 119.
13. 95.
15. -154.
17. 31.
19. 23.
21. 61.
23. -54.
25. -90.
27. 388.

EXERCISE 8.8.3

1. 10.
3. 35.
5. 23.
7. 61.
9. -54.
11. 60.
13. 4.
15. 0.
17. 2.
19. 14.
21. 5.
23. -3.
25. 5.
27. 2.
29. b.
31. $2p - q$.
33. $a + b$.
35. $5 + 5x$.
37. $2x + 2$.
39. $4q$.
41. $2q$.
43. $2r$.
45. 8.
47. $4 + t$.
49. -8.
51. $3x^4 + 7x^3 - x^2 - 2x - 3$.
53. $3x^3 + 3x^2 + 2x - 3$.
55. $x^5 + 3x^4 - 2x^3 + x^2 - 4x + 6$.

EXERCISE 9.1.6

1. -15.
3. $+14$.
5. -19.
7. 12.
9. -1.
11. -1.
13. 0.
15. -72.
17. -16.
19. 0.
21. 75.
23. -448.
25. $x^2 - xy$.
27. $p^4 - p^3q + p^2q^2$.
29. $x^4 + x^3y + x^2y^2$.
31. $-a^5 + a^4b - a^3b^2$.
33. $p^6 - p^5q + p^4q^2$.
35. $-t^8 + 2t^7 - t^6$.
37. $t^8 - 2t^7 + t^6$.
39. $x^5y^3 - 3x^4y^4 + 3x^3y^5 - x^2y^6$.
41. $a - 10b$.
43. $-4st + us - 5ut$.
45. $-9 + 6a - 4b$.
47. $-x^2 + 10x$.
49. $-16 + 36a - 27k$.
51. $-4ab$.
53. $p^2q - p^2r - q^2r + q^2p + r^2p - r^2q$.
55. $-8p^4 + 10p^3 - 6p^2 + 8p$.
57. $a^7b + a^6b^2 - a^5b^3 + a^4b^4 - a^3b^5 + a^2b^6$.
59. $-8x + 4$.
61. $10x^2 + 14x - x^3$.
63. $-3x^4 + 6x^7 - 3x^6$.
65. $xy(x - y)$.
67. $ab^2c(c^2 - a^2)$.
69. $x^2y^2(3 - 5xy + 4x^2yz)$.
71. $9pq^2r(r - 3p^2 + 9qr^2)$.
73. $25b^3(ab^5c^3 - 3a^2b^2 - 5c^3)$.
75. $(u - v)(u^2 + v^2)[(u - v) - (u^2 + v^2)]$.
77. $(x - y)^2$.

EXERCISE 10.2.2

1. 15.
3. 66.
5. 55.
7. -62.
9. 24.
11. -8.
13. 9.
15. -7.
17. 16.
19. 38.
21. -16.
23. -32.
25. 1247.
27. -5.

29. $(x^2 - y^2) \div (x - y) = x + y$.
31. $[(a - b)^5(a + b)^5] \div (a^2 - b^2)^5 = 1$.

49. -4.
51. 5.
53. -7.
55. -5.
57. 2.
59. 2.
61. 2.
63. -3.
65. 2.
67. 2.
69. 2.
71. 2.

73. 28, 14.
75. 5.
77. \$8, \$4, \$64, \$20.
79. 24′, 72′.
81. 198, 256.
83. 52, 26, 10, 4.

Exercise 10.3.4

1. 3.
3. 7.
5. 0.
7. $+2, -2$.
9. None.
11. $+1, -1$.
13. -2.
15. 1.
17. 9.
19. $+3, -3$.

Exercise 11.2.7

1. 9.
3. 24.
5. 2.
7. 3.
9. 3.
11. 15.
13. 48.
15. $\frac{1}{5}$.
17. $\frac{5}{8}$.
19. $\frac{1}{11}$.
21. $\frac{8}{7}$.
23. $\frac{3}{5}$.
25. $\frac{1}{1}$.
27. $1/x^2$.
29. y/x.
31. x^2/yz.
33. $3b/4a$.
35. $2x/1$.
37. $\frac{1}{1}$.
39. $\frac{405}{32}$.

41. $1/x$.
43. a/b.
45. $(x + y)/(x - y)$.

Exercise 11.3.5

3. $2 \cdot 5$.
5. $2^3 \cdot 5$.
7. $2^3 \cdot 3^2$.
9. $2 \cdot 3^4$.
11. $2^3 \cdot 3^4$.
13. $2 \cdot 3^3 \cdot 5$.
15. $2 \cdot 7^2$.
17. $2 \cdot 5^2 \cdot 7^2$.

Exercise 11.4.3

1. 15.
3. 10.
5. 14.
7. 210.
9. 5.
11. 245.
13. xy.
15. $11pq^2r^2$.
17. $\frac{3}{5}$.
19. $\frac{3}{2}$.
21. $\frac{3}{5}$.
23. $\frac{9}{74}$.

25. $\frac{38}{83}$.
27. $1/x^2$.
29. $(y + x)/(1 + x)$.
31. $(x + y)/(x - y)$.

Exercise 11.5.3

1. 60.
3. 180.
5. 9450.
7. 9450.
9. 420.
11. 18,900.
13. 108,000.
15. xy.
17. x^2y.
19. x^3y^3.
21. x^5y^5.
23. $60x^2y^2$.

25. $180x^3y^5$.
27. $23{,}100x^3y^3$.
29. $6x^2(3x + 4y)(5x + 7y)$.

EXERCISE 11.6.1

1. $\frac{3}{60}, \frac{50}{60}$.
3. $\frac{42}{4725}, \frac{40}{4725}$.
5. $\frac{21}{60}, \frac{50}{60}, \frac{124}{60}$.
7. $\frac{14}{24}, \frac{25}{24}$.
9. $\frac{3}{2}, \frac{3}{2}, \frac{1}{2}$.
11. $y/xy, x/xy$.
13. $y^2/xy, x^2/xy$.
15. $4y^2/6xy, 9x^2/6xy$.
17. $6y/3x^2, 7y/3x^2$.
19. $315x^2y/1800x^4y^2, 162/1800x^4y^2, 620x/1800x^4y^2$.
21. $15(x + y)/18(x + y)^2, 14/18(x + y)^2$.
23. $3x/(3x + 4y), 2y/(3x + 4y)$.

EXERCISE 11.7.4

1. $\frac{8}{15}$.
3. $\frac{2}{15}$.
5. $\frac{5}{7}$.
7. $\frac{15}{14}$.
9. $\frac{1}{2}$.
11. $\frac{5}{6}$.
13. $\frac{1}{1}$.
15. $\frac{49}{16}$.
17. $\frac{1}{3}$.
19. $\frac{1}{4}$.
21. $\frac{9}{100}$.
23. x^2/y^2.
25. x^3/y^3.
27. x^4/y^2.
29. x^6/y^9.
31. x^{10}/y^{10}.
33. $4a^2/15d^2$.
35. $\frac{1}{1}$.
37. $3x^2/4y$.
39. $3x^2/5a$.

EXERCISE 11.9.4

5. $\frac{1}{5}$.
7. $\frac{9}{35}$.
9. $\frac{8}{55}$.
11. $\frac{9}{35}$.
13. $\frac{9}{5}$.
15. $\frac{5}{9}$.
17. $\frac{3}{8}$.
19. $\frac{24}{25}$.
21. $\frac{18}{25}$.
23. 6.
25. $\frac{2}{5}$.
27. x^2/y^2.
29. x^3/y^3.
31. x/y^3.
33. $a/3bc$.
35. $ad^4/3bc^3$.
37. $4ab^2/3mc$.
39. $\frac{1}{6}$.
41. $\frac{3}{5}$.
43. 5.
45. 5.
47. $\frac{1}{2}$.
49. $\frac{31}{5}$.
51. $\frac{17}{5}$.
53. 2.
55. $\frac{59}{5}$.
57. $\frac{19}{5}$.
59. $\frac{12}{7}$.
61. $\frac{95}{121}$.
63. $\frac{7}{3}, \frac{9}{5}$.
65. $\frac{11}{2}$, 5.
67. $\frac{3}{2}, \frac{12}{7}$.
69. $\frac{3}{5}$, 5, $\frac{7}{2}$.
71. $\frac{13}{4}$.
73. 5.
75. 3.
77. $\frac{3}{5}$.
79. 2.
81. 1.
83. None.
85. 3, −3.
87. 9.
89. 4.

EXERCISE 11.10.1

1. 21, yes.
3. $\frac{37}{2}$, yes.
5. $\frac{21}{2}$, no.
7. 14, yes.
9. $\frac{32}{3}$ ft, yes.
11. 22, 11, 26, yes.
13. $\frac{74}{5}$ in., $\frac{37}{5}$ in., $\frac{94}{5}$ in., yes.
15. $\frac{74}{5}, \frac{37}{5}, \frac{94}{5}$, yes.
17. 50°, 100°, yes.
19. 11 ft by 22 ft, yes.
21. $\frac{10}{3}$ ft, yes.
23. 108°, 54°, 18°, yes.
25. $\frac{1440}{11}, \frac{360}{11}, \frac{180}{11}$, no.

EXERCISE 12.1.6

1. $\frac{1}{2}; \frac{a+1}{b}$.
3. $8; \frac{p+r}{q}$.
5. $\frac{8}{7}; 8/x$.
7. $\frac{49}{25}; 7a/5b$.
9. $\frac{12}{5}; 2a/b$.
11. $\frac{3}{5}pq$.
13. $\frac{17}{20}; \frac{1+a^2}{ax}$.
15. $\frac{8}{3}; \frac{a+ab}{b}$.
17. $\frac{17}{8}, \frac{a^2+9}{b^4}$.
19. $\frac{4+a^2}{2a}$.
21. $\frac{3a+2b}{30}$.
23. $\frac{3a+2b}{6x}$.
25. $\frac{3xy+4x}{14y^2}$.
27. $\frac{20py+5qx}{12xy}$.
29. $\frac{xy+3y+x^2+x}{x^2y^2}$.
31. 35.
33. $\frac{40}{11}$.
35. $\frac{a^2+a^2b+ab^2+ab^3}{b^2}$.
37. $\frac{c^2x^2+2cdx+d^2}{x^2}$.
39. $\frac{2x+y}{y^2}$.
41. d/b.
43. $\frac{y^2+3y+10}{y^2+y+20}$.
45. $\frac{b+a}{b}$.
47. $\frac{b+a}{b}$.
49. d/b.
51. -2.
53. $0, -1, -2$.
55. $2, -2$.
57. $3, 2$.

EXERCISE 12.2.3

1. $\frac{10}{7}$.
3. $\frac{2}{3}$.
5. 3.
7. $\frac{xy+y}{xy+x}$.
9. y/x.
11. $\frac{3x+30}{31x+6}$.
13. $\frac{2x^2+5x+3}{x+2}$.
15. $\frac{a^3+2a^2+7a+8}{a^3+7a}$.
17. $\frac{x^2y^2(x+y)}{x^4+y^4}$.

EXERCISE 12.5.2

1. $\frac{2}{3}$.
3. $\frac{1}{3}$.
5. $4x^2/3$.
7. $-\frac{3}{2}$.
9. $-\frac{1}{5}$.
11. $-\frac{15}{14}$.
13. $\frac{24}{7}$.
15. $-\frac{3}{2}$.
17. -1.
19. $a/6b$.
21. $-3/2y^2$.
23. $-3x^2/5a$.

EXERCISE 12.6.2

1. $\frac{3}{8}$.
3. -5.
5. $\frac{3}{4}$.
7. $-\frac{17}{8}$.
9. $(3-2a)/x$.
11. $(4r-3s)/24$.
13. $31t/12$.
15. $(11a+b)/20$.
17. $\frac{-2b+5a}{ab}$.
19. $\frac{3x-2}{x^2}$.
21. $\frac{c+2a-3b}{abc}$.
23. $\frac{y^2-xy+3x^2}{x^2y^2}$.

25. $\frac{-5y + 4x}{x^2y^2}$.

27. $\frac{3n^2 + 5mn - 2m^2}{6mn}$.

29. $\frac{15 + x - 4x^2}{6x^2}$.

31. $\frac{1}{2x + 2}$.

33. $\frac{-39}{40(a + b)}$.

35. $\frac{p - 5q}{p^2 - q^2}$.

37. $\frac{-1}{(a + 2)(a + 3)}$.

39. $\frac{2a - 3}{a}$.

41. $\frac{3x^2 - 3x + 2}{3x}$.

43. $\frac{-4y + 7}{2y - 1}$.

45. $\frac{x}{x - 2y}$.

47. $\frac{2(x^2 - 2)}{x^2 - 1}$.

49. x.

51. $\frac{x^2 - y^2}{y^2}$.

53. $\frac{p^2 + q^2}{p^2 - q^2}$.

55. $\frac{a^2 + a - 1}{a^2 - a + 1}$.

57. $\frac{1}{x - y}$.

59. 2.

61. $\frac{142}{105}$.

63. $-\frac{3x + 4}{12}$.

65. $\frac{6x^3 - 21x^2 - 11x + 2}{12}$.

67. $-\frac{5x^2(2x - 3)}{2}$.

69. x^5/y^5.
71. $1/a^4$.
73. x^2/y^4.
75. $1/b^2$.
77. y^2/x^2.

Exercise 12.7.1

1. $\frac{7}{9}$.
3. $\frac{71}{40}$.
5. $\frac{31}{71}$.
7. $\frac{1}{1{,}000{,}000}$.
9. 0.
11. $-\frac{97}{250}$.
13. $-\frac{301}{180}$.

Exercise 12.8.4

1. x^5/y^5.
3. $(a/b^2)^t$.
5. 1.
7. b^4/a^3.
9. $\frac{1}{2ax^5}$.
11. q^2r^3/p.
13. $\frac{1}{(x - y)^2}$.
15. $1/r^{p+1}$.
17. $1/2^n$.
19. $\frac{b^{2s}}{a^s}$.
21. x^s.
23. $\frac{-8b^n}{a^m}$.
25. $\frac{x^2 - y^2}{x^2y^2}$.
27. $a^4b^2 + a^2b^4$.
29. $1/x$.
31. x.

Exercise 13.2.2

1. $3x^4 - 6x^3y + 9x^2y^2$.
3. $2^8a^5b + 2^7a^4b^2 - 2^6a^3b^3 + 2^5a^3b^4$.

5. $x^2 - y^2$.
7. $4r^2 - 9s^2$.
9. $a^4 - b^4$.
11. $9p^2q^2 - 1$.
13. $x^4 - y^6$.
15. $9a^6 - 4b^6$.
17. $x^2 - 2x + 1$.
19. $x^2 - 6x + 9$.
21. $4p^2 - 4pq + q^2$.
23. $9s^2 + 12st + 4t^2$.
25. $1 + 4ab + 4a^2b^2$.
27. $x^4 + 2x^2y + y^2$.

29. $4u^2 - 12uv^2 + 9v^4$.
31. $16p^4 - 24p^2q^2 + 9q^4$.
33. $4a^2b^4 - 12a^3b^3 + 9a^4b^2$.
35. $x^2 + x - 2$.
37. $a^2 - 5a - 14$.
39. $p^2 - 8p + 15$.
41. $2r^2 + 5r + 2$.
43. $6b^2 - 25bc + 25c^2$.
45. $x^4 + 5x^2y + 6y^2$.
47. $2u^4 + 5u^2v^2 + 3v^4$.
49. $x^2 - y^2 - 2yz - z^2$.
51. $a^2 + 2ab + b^2 - c^2$.

53. $p^2 + 2pq + q^2 + 2pr + 2qr + r^2$.
55. $p^2 + q^2 + r^2 + 2pq + 2qr + 2pr$.
57. $p^2 + q^2 + r^2 - 2pq - 2qr + 2pr$.
59. $4x^2 + 12x - 8xy + 9 - 12y + 4y^2$.
61. $9x^2 + y^2 + 4z^2 + 6xy + 4yz + 12xz$.

63. $a^2 + 2ab + b^2 - 4c^2$.
65. $a^2 + 2ab + b^2 - c^2 - 2cd - d^2$.
67. $a^2 + 2ab + b^2 - c^2 + 2cd - d^2$.
69. $16x^4 - 81$.
71. $x^4 + 2x^3y + x^2y^2 - y^4$.

Exercise 13.4.2

1. $5x^2 - 9x + 5$.
3. $3x^3 + 6x - 10$.
5. $3x^3 + 7x + 4$.
7. $3x^2 + 4$.
9. $-4x^2 + 24x + 9$.
11. $-4x^2 - 5$.
13. $30x^4 - 20x^3 - 100x^2 + 5x$.
15. 2, 2 in x, 2 in y.
17. 3, 2 in x, 2 in y.
19. 4, 3 in x, 3 in y.

Exercise 14.2.5

1. $3x^2(2x^2 + 3)$.
3. $3x(3 - x)$.
5. $15a^2x^2(x^2 + 2a)$.
7. $(x + y)(3 + 2x)$.
9. $(a + b)^2$.
11. $(x + y)^2(x^2 + y^2)$.
13. $9pq^2(r^2 + 3p^2 - 9qr^3)$.
15. $4r^2s^2t^2(2^5 + 2^3r^3t + r)$.
17. $x^2(x + 3)(x - 3)$.
19. $(4x + 1)(4x - 1)$.
21. $3n^2(3m + 2n)(3m - 2n)$.
23. $(x + y)(x - y)(x^2 + y^2)$.
25. $(a + 2b)(a - 2b)(a^2 + 4b^2)$.
27. $(x - 4)^2$.
29. $(2a + 3)^2$.
31. Not factorable.
33. $(5c + 2d)^3$.
35. $(x + y)^2(x - y)^2$.
37. $x^2(x - 4)^2$.
39. $b^2(3a - 2)^2$.
41. $16(5x + 3)^2$.
43. $4y^2(5x + 3y)^2$.
45. $(x + 3)(x + 4)$.
47. $(x + 12)(x + 1)$.
49. $(x - 6)(x - 2)$.
51. $x(x + 7)$.
53. $(x + 2)(x + 5)$.
55. $(x - 2)(x + 9)$.
57. Not factorable.
59. $(x - 10)(x + 3)$.
61. Not factorable.
63. $(r + 4)(r - 5)$.
65. $(c + 15)(c - 5)$.
67. $(y - 15x)(y - 3x)$.
69. $9(a - 3)(a - 4)$.
71. $x^2(x^2 - 11x + 12)$.
73. $5(x + 5)(x - 5)(x^2 + 1)$.
75. $(x + 3)(x - 3)(x + 2)(x - 2)$.
77. $(3x - 2)(x - 2)$.
79. $(3x - 2)(2x + 3)$.
81. $(4x - 3)(3x - 5)$.
83. Not factorable.

85. $15(x-1)^2$.
87. Not factorable.
89. $-2(4x-3)(2x+3)$.
91. $-(7x-3)(3x-7)$.
93. $(x+y+a)(x+y-a)$.
95. $(x+y+a)(x+y-a)$.
97. $(a^2-b^2+c)(a^2-b^2-c)$.
99. $(a+x-y)(a-x+y)$.

101. $(3x+2)(x-1)(3x-2)(x+1)$.
103. $(3x+2)(2x-3)(3x-2)(2x+3)$.

105. $2, -2$.
107. $2, -2$.
109. -3.
111. $\frac{9}{5}, \frac{7}{3}$.
113. $2, -2, 3, -3$.
115. $-\frac{3}{5}, 1, \frac{3}{5}, -1$.
117. $-\frac{2}{5}, 5, \frac{2}{5}, -5$.
119. $\frac{3}{5}, -\frac{3}{5}, 1, -1$.
121. $\frac{2}{5}, -\frac{2}{5}, 5, -5$.
123. $0, \frac{4}{9}, \frac{9}{4}$.

EXERCISE 14.3.2

1. $3x^2$; $6x^3(x^2-1)$.
3. p^2q^2; $p^2q^2(q^2-p^2)$.
5. $p^2q^2(q+p)$; $p^2q^2(p^2-q^2)$.
7. $3a+7b$; $(3a-7b)(3a+7b)^2$.
9. x^2+4; $x^2(x^4-16)$.
11. $x-2$; $(x-2)(x+2)(x+1)$.
13. $m+3$; $(m+3)^2(m+7)$.
15. $b^2(3a-2)$; $b^4(3a+2)(3a-2)^2$.

17. $6x-5$; $(36x^2-25)(2x+3)$.
19. $4x^2-1$; $(4x^2-1)(x^2-4)(x^2-16)$.

21. $7/x$.
23. $3/2y$.
25. $\dfrac{x-1}{2(x-2)}$.
27. $\dfrac{a-1}{a+2}$.
29. $\dfrac{3x+5}{x-2}$.
31. $\dfrac{x^2-4}{x^2}$.
33. $\dfrac{2(2x+3)}{x(6x+5)}$.
35. $\dfrac{(2x+3)(3x-2)}{(2x-3)(3x+2)}$.
37. $\dfrac{3(x+1)}{3x+4}$.
39. $\dfrac{x+3}{x-3}$.
41. $3x^2/4y$.
43. $(a+2)/(a-1)$.
45. p.
47. $(x+5)/(x+3)$.
49. $(x-3)/(x+7)$.
51. $4x-7$.
53. $\dfrac{5}{2(x+2)}$.
55. $\dfrac{2(3x^2+11x+9)}{(x+1)(x+2)(x+3)}$.
57. $\dfrac{2x}{(x+y)^2(x-y)}$.
59. $2/(p+q)$.
61. $\dfrac{2}{(x-1)(x-3)}$.
63. $\dfrac{1}{(x^2-1)(x+2)}$.
65. $\dfrac{x}{(x^2-y^2)(x+2y)}$.
67. $\dfrac{1}{x+3}$.
69. $\dfrac{-17}{15(x-1)}$.
71. $\dfrac{2x-1}{x^2-4}$.
73. $\dfrac{2}{1-x^2}$.
75. $\dfrac{2x}{(x-1)^2}$.
77. $-2x$.
79. b^2.

EXERCISE 14.4.4

1. $(x - 1)(x^2 + x + 1)$.
3. $(2a + 1)(4a^2 - 2a + 1)$.
5. $(a + 2)(a^2 - 2a + 4)$.
7. $(3x + 2)(9x^2 - 6x + 4)$.
9. $(2p + 5q)(4p^2 - 10pq + 25q^2)$.
11. $(xy - 1)(x^2y^2 + xy + 1)$.
13. $(3ab - 5c)(9a^2b^2 + 15abc + 25c^2)$.
15. $(2x + 1)(x^2 + x + 1)$.
17. $2(3x^2 + 1)$.
19. $(x + 1)(7x^2 + 5x + 1)$.
21. $5(x + 1)(7x^2 + 11x + 7)$.
23. $-(x + 5)(19x^2 + 100x + 133)$.
25. $(x - y)(x^2 + xy + y^2)(x + y)(x^2 - xy + y^2)$.
27. $x^3(x - 4)(x^2 + 4x + 16)$.
29. $(2x - 1)(4x^2 + 2x + 1)(x + 2)(x^2 - 2x + 4)$.
31. $x^3(2x - 1)(4x^2 + 2x + 1)$.
33. $9x^5(x - 2)(x^2 + 2x + 4)$.

EXERCISE 14.5.1

1. $(a - b)(x + y)$.
3. $(a - b)(x - y)$.
5. $(x + 2)(x^2 + 1)$.
7. $(x + 2)(x + 1)(x - 1)$.
9. $(x - 3)(2x + 3)(2x - 3)$.
11. $(x - 2 + y)(x - 2 - y)$.
13. $x(3x + 1 + 2y)(3x + 1 - 2y)$.
15. $(x^2 + y^2 + xy)(x^2 + y^2 - xy)$.
17. $(x^2 + 8y^2 + 4xy)(x^2 + 8y^2 - 4xy)$.
19. $(x + a)(x + 1)(x^2 - x + 1)$.
21. $(x + a)^2(x^2 - ax + a^2)$.
23. $(x + a)(x - a)(x^2 + ax + a^2)$.

EXERCISE 14.6.1

1. $(x^2 + 4 + 2x)(x^2 + 4 - 2x)$.
3. $(4x^2 + 1 + 2x)(4x^2 + 1 - 2x)$.
5. $x^2(4x + 1)^2$.
7. $(x^2 + 2 + 2x)(x^2 + 2 - 2x)$.
9. $(x^2 - 3 + x)(x^2 - 3 - x)$.
11. $(x^2 + 4 + 3x)(x^2 + 4 - 3x)$.
13. $(3x^2 + 4y^2 - 4xy)(3x^2 + 4y^2 + 4xy)$.
15. $(x - 2y)(x^2 + 2xy + 4y^2)(x + 2y)(x^2 - 2xy + 4y^2)$.
17. $(3x - 2)(9x^2 + 6x + 4)(3x + 2)(9x^2 - 6x + 4)$.

EXERCISE 15.2.1

1. $2, -4, 0, 18, 0$.
3. $4, 0, 2, \frac{2}{3}a + 2, \frac{2}{3}b + 2$.
5. $12, 0, 0, x^2 + 2x - 3, x^2 + 4x$.
7. 6.

EXERCISE 15.5.2

15. $(2, 5)$.
17. $(1, 7)$.
19. $(-2, -2)$.

EXERCISE 16.1.2

1. $7x^2 + 4x - 4$.
3. $10x^3 - 6x^2 + 3x$.
5. $4x^4 - 2x^3y - 2x^2y^2 + 9xy^3 - 3y^4$.

7. $-5x^4 + 4x^3 - 11x^2 - 2x + 19$.
9. $x^4 - x^3y - x^2y^2 - 8xy^3$.
11. $x^5 + x^4 + x^2 - x$.
13. $x^5 + 7x^4 + 2x^3 + 3x^2 - 5x + 9$.
15. $10x^5 - 7x^4 + 5x^3 + 65x^2 - 55x - 18$.
17. $2x^7 + 3x^5 + 2x^4 - 12x^3 + 5x^2 + 7x - 7$.

EXERCISE 16.2.1

1. $x^3 + 3x^2 + 3x + 1$.
3. $15x^5 + 9x^4 + 2x^3 - 27x^2 - 8x + 14$.
5. $4x^9 + 5x^7 + 27x^5 - 11x^3 + 35x$.
7. $x^3 + 6x^2 + 12x + 8$.
9. $27x^3 - 54x^2 + 36x - 8$.
11. $x^3 + 3ax^2 + 3a^2x + a^3$.
13. $x^4 + 4ax^3 + 6a^2x^2 + 4a^3x + a^4$.

EXERCISE 16.3.7

15. 1, 2, -1.
17. $\frac{7}{5}$, $\frac{3}{2}$, $-\frac{3}{2}$.

EXERCISE 16.4.4

1. $6x + 5$.
3. $x^2 + 2x + 2$.
5. $x^3 + 3x^2 + 2x$.
7. $x + 5$.
9. $x^2 + 4x + 3$.
11. $9x + 7$.
13. $2x^3 + 4x^2 + 3x + 8$.
15. $2x^3 + 7x^2 + 4x + 4$.
17. $2x + 3$.
19. $2x + 7$.

EXERCISE 17.1.4

1. $3/x^2$.
3. 5.
5. $1/x^2$.
7. $-1/27x^6$.
9. $1/a^3$.
11. $-3/x^2y^2z^3$.
13. $8/a^7$.
15. $\dfrac{1}{a} + \dfrac{1}{b} = \dfrac{b + a}{ab}$.
17. $1 + \dfrac{a}{b} = \dfrac{b + a}{b}$.
19. $\dfrac{1 + a^2}{ab}$.
21. $-ab$.
23. $\dfrac{x^2y^2}{y^2 - x^2}$.
25. $\dfrac{e^{2x} + 1}{e^{2x} - 1}$.
27. $\dfrac{3}{10^4}$.
29. $3 \cdot 10^4$.
31. $\frac{35}{2}$.

EXERCISE 17.2.2

15. 0.5.
17. 0.575.
19. 0.285714 285714 285714 $\cdots$.

EXERCISE 17.3.1

1. $\frac{2}{3}$. 3. $\frac{10}{99}$. 5. $\frac{23}{9900}$. 7. $\frac{60}{7}$.

EXERCISE 17.4.4

3. $(3.2)\cdot 10^7$. 5. $(6.01)\cdot 10^{13}$. 7. $(3.5)\cdot 10^{-5}$.
9. $(6.74)\cdot 10^0$. 11. 3,600,000,000. 13. 0.000041.

15. 0.000 000 000 000 6214. 17. $(6.0)10^{-7}$.
19. $(8.69)10^{-25}$ lbs, $(1.15)10^{24}$.

EXERCISE 18.1.6

1. Identity; none. 3. Identity; none.
5. Conditional. 7. Conditional.
9. Identity; 2 and 3. 11. Identity; 1 and -1.
13. Identity; none.

EXERCISE 18.2.6

1. 5. 3. -2. 5. 4. 7. $-\frac{9}{2}$.
9. $\frac{59}{22}$. 11. 0. 13. $-\frac{13}{4}$. 15. 6.
17. $\frac{3}{8}$. 19. 5. 21. 2, 3. 23. 2.
25. No solution. 27. 2, -4.

EXERCISE 18.3.2

1. 2. 3. 5. 5. $-\frac{11}{5}$. 7. -1.
9. No solution. 11. 2. 13. -13. 15. -1.
17. -9. 19. -5. 21. 1. 23. 2.
25. -3. 27. 2. 29. 0. 31. 9.
33. No solution. 35. No solution. 37. No solution. 39. 1.
41. -5. 43. No solution. 45. 7. 47. 4, -4.
49. 1. 51. $-\frac{2}{3}$, $\frac{3}{2}$. 53. No solution. 55. 2, $\frac{1}{2}$.

57. $\frac{5}{4}$, $-\frac{5}{4}$. 59. 0, 7, -4.
61. $\frac{2}{3}$, 1, $-\frac{2}{3}$, -1. 63. 2, -2, 1, -1.
65. 70, 45. 67. No solution.
69. 25 lbs @ 80¢, 50 lbs @ 95¢. 71. 30 and 50 mph.
73. \$500, \$850, \$1250, \$2700. 75. 3 inches.

77. 8 and 20 or -26 and -14.
79. 12 noon. They also were 13 inches apart $\frac{23}{41}$ of an hour before 9 o'clock.
81. 12 inches.
83. $\frac{2}{9}$ or $-\frac{9}{2}$.
85. 24 ft at the end of 1 second and $\frac{3}{2}$ seconds; level with the roof at the end of $\frac{5}{2}$ seconds; strikes the ground at the end of 5 seconds.

87. 5 inches. 89. 27. 91. 75. 93. 12 feet.
95. 120.

EXERCISE 18.4.5

1. 2.
3. $\frac{7}{2}$.
5. 2, -1.
7. $\frac{5}{2}$, $-\frac{5}{3}$.
9. 3.4, 0.6.
11. 2.
13. No solution.

EXERCISE 18.5.1

1. $(\frac{3}{2}, \frac{3}{4})$.
3. $(-\frac{5}{3}, 3)$.
5. $(-\frac{1}{3}, \frac{7}{3})$.
7. $(\frac{35}{6}, -\frac{7}{6})$.
9. $(1, 0)$.
11. No solution.
13. $(1, 2)$, $(-2, -1)$.
15. $(4, -1)$, $(-1, 4)$.

EXERCISE 18.5.2

1. 5, 7.
3. 8, 15.
5. 24.
7. 55 at \$20, 45 at \$30.
9. 4 by 10 feet.
11. 9 and 5, -4 and -8.
13. 20 and 21.

EXERCISE 18.5.4

1. $(2, 1, 1)$.
3. $(-2, -3, 4)$
5. 22, 7, and 12 gals.
7. 4, -4, 6.

EXERCISE 19.1.2

1. 1.732.
3. 1.260.
7. Then $x = y - 1$ is rational; but x is irrational.

EXERCISE 19.2.4

21. 0.
23. $\frac{1}{2}$.
25. $\frac{1}{2}$.
27. $\frac{4}{5}$.
29. 6.

EXERCISE 19.3.1

1. 2.646.
3. 5.292.
5. 6.928.
7. 3.420.
9. 22.36.
11. 7.071.
13. 12.60.
15. 4.378.
17. 2.449.
19. 0.707.
21. 1.202.
23. 1.816.

EXERCISE 19.5.4

1. $\sqrt{2}$, $\sqrt[3]{3^2} = \sqrt[3]{9}$, 2, 3, $\sqrt{(17)^5}$.
3. 27, 27, 16, 16.
5. 3, 5, 4, 729.
7. $\sqrt{a+b}$, $\sqrt{a-b}$.
9. $\sqrt[10]{10^3}$, $\sqrt[10]{10^5} = \sqrt{10}$, $\sqrt[1000]{10^{301}}$, $\sqrt[1000]{10^{477}}$, $\sqrt[10,000]{10^{8451}}$.
11. $c^{1/2}$.
13. $t^{3/2}$.
15. $3x^{1/2}$.
17. $3x^{1/2}$.
19. $p + q$ if $p + q \geqq 0$; $-(p + q)$ if $p + q < 0$.

21. $3(x^2 - y^2)^{1/2}$.
23. $b^{7/3}$.
25. $a^{1/2}b^{3/4}$.
27. x^2.
29. $x^{10}y^6$.
31. $x^{5/6}$.
33. $c^{3/2}$.
35. $t^{11/15}$.
37. b.
39. $c^{1/6}$.
41. $1/x^{1/6}$.
43. $1/x^{7/6}$.
45. $1/b^{5/6}$.
47. 10.
49. 10.
51. 2.
53. 4.
55. $\frac{5}{2}$.
57. $\frac{1}{729}$.
59. 5400.
61. -0.2.
63. 3.1.
65. 0.227.
67. $\frac{7}{12}$.
69. -9.
71. 9.
73. 0.
75. 0.008.

Exercise 19.5.6

1. $2\sqrt{2}$.
3. $7\sqrt{2}$.
5. $7\sqrt{2} - 2\sqrt{3}$.
7. $60\sqrt{2} + 50\sqrt{3}$.
9. $50\sqrt[3]{4} - 50\sqrt[3]{2}$.
11. 0.
13. $3x\sqrt{x}$.
15. $ts^2\sqrt{5ts}$.
17. $t\sqrt[3]{5x^2}$.
19. $6a^2b\sqrt{a}$.
21. $\sqrt{18}$.
23. $\sqrt{4x^3y^4}$.
25. $\sqrt[3]{5s^2t^3}$.

Exercise 19.6.2

1. $\sqrt{30}$.
3. $6\sqrt{2}$.
5. $3a\sqrt{5}$.
7. $3\sqrt[3]{2}$.
9. $-3x^2$.
11. $\sqrt[6]{3^5}$.
13. $2a^3 \sqrt[12]{2^4 \cdot 3^4 \cdot a^7}$.
15. $3/\sqrt{5}$.
17. $3\sqrt{3}$.
19. $6/\sqrt{5}$.
21. $\sqrt[3]{\frac{4}{25}}$.
23. $\frac{1}{3}\sqrt[3]{5x^2/4}$.
25. $3x\sqrt{2x/5}$.
27. 7.
29. $x^2 - 2$.
31. 2.
33. $x^2 - 4x + 1$.
35. $x^2 - 2x + \frac{1}{4}$.
37. $4x^2 + 4\sqrt{5}\,x + 5$.

Exercise 19.7.2

1. $\dfrac{2\sqrt{3}}{3}$, 1.155.
3. $\dfrac{\sqrt[3]{9}}{3}$, 0.693.
5. $\sqrt[3]{4}$, 1.587.
7. $\dfrac{2\sqrt{3}}{3}$, 1.155.
9. $\dfrac{6 + 2\sqrt{2}}{7}$, 1.261.
11. $2 + \sqrt{2}$, 3.414.
13. $\sqrt{6} - \sqrt{2}$, 1.035.
15. $5 + 2\sqrt{6}$, 9.898.
17. $\dfrac{-6 + 7\sqrt{6}}{6}$, 1.857
19. $\sqrt{2a}/2a$.
21. $3x\sqrt[3]{2x^2}$.
23. $\sqrt{30st}/6s$.
25. $\sqrt{x^2 - xy}/x$.
27. $\sqrt[3]{xy}/y$.
29. $\dfrac{x + 2y + 2\sqrt{2xy}}{x - 2y}$.
31. 5.831.
33. 38.2 ft.

Exercise 20.2.6

1. $2i$.
3. $2i\sqrt{2}$.
5. $5i\sqrt{2}$.
7. $3 + 3i$.
9. $5 + 8i\sqrt{2}$.
11. $i, -1, -i, 1$.
13. $i, -1, -i, 1, -i$.
15. $1 - 2i$.
17. $3 + (3 + \sqrt{2})i$.

19. $(3\sqrt{2} - 5) + (5 - 3\sqrt{2})i$.

21. 52.
23. $-5 - 12i$.
25. $12 - 5i$.
27. $-2 + 3i$.
29. $2 - 2i$.
31. 8.
35. $(\frac{2}{3}, \frac{3}{2})$.
37. $(3, 4)$.

Exercise 20.3.3

1. $-i$.
3. $-i$.
5. $\frac{8}{5} + 4i/5$.
7. $-\frac{2}{3} - 5i/3$.
9. $(-3, 4)$.
11. $\frac{5}{13} + 12i/13$.
13. $-\frac{1}{6}$.
15. $-\frac{5}{13}$.

Index

The numbers refer to pages. References to definitions and postulates are in bold-face type.

Abscissa, 171
Absolute value, **59,** 60
Addition, associative law for, **8,** 61, 121, 136
  commutative law for, **6,** 44, 61, 136
  of complex numbers, 245, 246
  of fractions, with different denominators, **116**
    with same denominator, **115**
  grouping in, 9, 10, 62
  of integers, 56, 57, 59, **60,** 62
    on number scale, 57
  inverse of, 35
  of natural numbers, **5, 6, 8,** 9, **20**
  of polynominals, 175
  postulate of closure for, **5,** 136
  of rational numbers, 128
  of signed numbers, 56, **60,** 62
Addition and multiplication, grouping in, 22
Addition and subtraction, grouping in, 68
Addition, subtraction, multiplication, division, grouping in, 82
Algebra, 2
Algebraic expression, integral rational, 141
  rational, 141
Algebraic irrational, 229
Approximation, decimal, 194, 227, 229, 232
  rational, 232, 254
Associative law, for addition, **8,** 61, 121, 136
  for multiplication, **13,** 121, 136, 250

Base, **25**
Binomial, 142
  quadratic, 145
Braces, 8
Brackets, 8

Cartesian coordinates, 171
Characteristic, **197**
Closure, law of, for addition of integers, 61
  postulate of, for addition, **5,** 136
    for multiplication, **12**
Coefficient, 33, 176
Common denominator, least, 102, 121
Common factor, 31, 145
  highest, 31, 95, **99,** 151
  taking out the, 21, 30
Common multiple, least, 81, **101,** 151
Commutative law, for addition, **6,** 44, 61, 136
  for multiplication, **12,** 44, 73, 136, 250
Complex fractions, 119, 120, 244
Complex imaginary number, 245
Complex number system, 244
Complex numbers, **244**
  addition of, 245, 246
  algebra of, 245
  conjugate, **245,** 247
  division of, **247**
  equality of, **245**
  imaginary part of, 245
  multiplication of, 245
  real part of, 245
  simplest form of, 245

Complex numbers, subtraction of, 246
Composite number, 96, 98
Conditional equation, **201**
Conjugate complex numbers, **245,** 247
Consecutive natural numbers, 18
Constants, literal, 140
Coordinates, 171
Cube of x, 26
Cube root, **230,** 254
  table of, 233
Cubes, sum and difference of, 151

Decimal approximation, 194, 227, 229, 232
Decimal fractions, 192
Decimal notation, 184, 196
Decimal polynomials, **185**
Decimals, repeating or periodic, 194, 195
  terminating, 194
Degree, of monomial, 142
  of polynomial, 142, 192
  of terms, 142
Denominator, **90,** 93, 94, 125
  least common (LCD) 102, 121
  rationalizing, 241, 247
  secondary, 121
Dependent variable, **165**
Difference, *see* Subtraction
  of two cubes, 151
  of two squares, 145, 160
Digits, **185,** 186
Directed numbers, 53
Distributive law, **20,** 121, 136
  extension of, 21, 136, 138
Dividend, **79,** 108, 180
Division, **79, 180**
  of complex numbers, **247**
  of fractions, **107,** 108
  of integers, 80, 110, **180**
  law of signs for, 81
  of natural numbers, **79,** 80, 104, 106, 249
  of polynomials, 182
  of powers, 132, 133, 137, 189, 190
  of radicals, 239
  of rational numbers, 127
  as repeated subtraction, 180, 182
Division, of signed numbers, 81, 126, 127
  by zero, 87, 106
Divisor, **79,** 108, 180
  of integer, **80,** 84, **180,** 251
  of polynomial, **182**

Equal to, greater than or, 48
Equal complex numbers, **245**
Equal fractions, **92,** 93, 121
Equals sign, **7**
Equations, 112, **200**
  conditional, **201**
  equivalent, 40, 49, **203,** 204, 205, 207, 250, 253, 254
  graphs of, 218
  identical, **200,** 201
  roots of, **202**
  setting up, 16, 17
  simultaneous, 221, 222, 225
  solutions of, 40, 63, 110, 112, **202,** 221, 225
  solving, 16, 204, 210
Equivalent equations, 40, 49, **203,** 204, 205, 207, 250, 253, 254
Even number, 6, 234, 250
Exponents, **25,** 197
  integer, **190**
  laws of, 28, 29, 132, 133, 137, 189, 235
  negative fractional, **235**
  negative integer, **190**
  rational, **235**
  zero, **190**
Extension of distributive law, 21

Factor, common monomial, 31, 145
  highest common (HCF), of natural numbers, 31, 95, **99**
    of polynomials, 151
  of integer, **80,** 84, **180,** 251
  of polynomial, **182**
Factoring, difference of two squares, 145
  fourth-degree polynomials, certain, 161
  by grouping, 160
  quadratic binomials, 145

Factoring, quadratic trinomials, 146
  sum and difference of two cubes, 158
Factors, **11,** 98, 237, 238, 251
  prime, of composite numbers, 98, 144, 228
    of polynomials, 144
First-degree functions, 173
Fourth-degree polynomials, factoring of certain, 161
Fractions, 89, **90,** 105, 230
  addition of, with different denominators, **116**
    with same denominator, **115**
  complex, 119, 120, 244
  decimal, 192
  division of, **107,** 108
  equal, **92,** 93, 121
  in lowest terms, 95, 153
  multiplication of, **104,** 105
  negative, 123, 124
  operations on, 94
  positive, 123, 124
  product of, **104,** 105
  quotient of, **107,** 108
  secondary, 121
  signed, 123
  simple, 120
  subtraction of, 128
Functional notation, 167
Functions, 163, **165**
  graphs of, **172**
  linear, 173
  values of, 167, 218

Generalization, 44, 52, 90, 123, 229, 243
Graphs, of equations, 218
  of first-degree functions, 173
  of functions, **172**
Greater than and less than, for integers, **71**
  for natural numbers, 37
  for rational numbers, 131
  for zero, 46
Greater than or equal to, 48
Grouping, in addition, 9, 10, 62
  in addition and multiplication, 22
Grouping, in addition and subtraction, 68
  in addition, subtraction, multiplication, and division, 82
  in factoring, 160
  in multiplication, 14
  in subtraction and multiplication, 38
  symbols of, 8

Highest common factor (HCF), 31, 95, **99**
  of polynomials, 151

i, 243
Identical equations, **200,** 201
Identity, **200,** 201
Imaginary number, complex, 245
  pure, **243,** 245, 254
Imaginary part of complex number, 245
Imaginary unit, **243**
Incorrect methods, 212
Independent variable, **165**
Index, of radical, 230, 239
Integer exponents, **190**
Integers, addition of, 56, 57, 59, **60,** 62
  division of, **80,** 110, **180**
  divisor of, **80,** 84, **180,** 251
  greater than and less than for, 71
  multiplication of, **73, 76**
    law of signs for, 76
  negative, 53
  positive, 53
    and natural numbers, 55
  set of, 54, 80, 89
  signed, 53
  subtraction of, 56, 57, 59, 63
Integral rational algebraic expression, 141
Inverse operations, 35, 81
Irrational number, 195, 228, **229,** 234
  algebraic, 229
  transcendantal, 229

Law of signs, for division, 81
  for multiplication, **73,** 74, **76,** 137

Laws of exponents, 28, 29, 132, 133, 137, 189, 235
Least common denominator (LCD), 102, 121
Least common multiple (LCM), 81, **101**
  of polynomials, 151
Less than, *see* Greater than and less than
Less than or equal to, *see* Greater than or equal to
Letters for numbers, 6, 15, 140
Like terms, 33
Limit, 251
Linear functions, 173
Literal constants, 140
Literal numbers, 6, 15, 140
Lowest terms, fractions in, 95, 153

Minuend, 37
Missing terms, 175, 186
Monomial, degree of, 142
Multiple, least common (LCM), 81, **101**
  of polynomials, 151
Multiple of, **80,** 84
Multiplication, associative law for, **13,** 121, 136, 250
  commutative law for, **12,** 44, 73, 136, 250
  of complex numbers, 245
  of fractions, **104,** 105
  grouping in, 14
  of integers, **73, 76**
  inverse of, 35, 81
  law of signs for, **73,** 74, **76,** 137
  of natural numbers, 10, **11, 12, 13, 20, 25**
  of polynomials, 177
  postulate of closure for, **12**
  of powers, 28, 29, 137, 189
  of radicals, 239
  of rational numbers, 126
  of signed numbers, 73, 76, 126
  symbols for, **11**
  by zero, 45
Multiplying out, 21

Natural numbers, addition of, **5, 6, 8,** 9, **20**
  consecutive, 18
  division of, **79,** 80, 104, 106, 249
  greater than and less than for, 37
  multiplication of, 10, **11, 12, 13, 20, 25**
  and positive integers, 55
  subtraction of, **36,** 37, 44, 52, 63
Negative exponents, fractional, 235
  integer, **190**
Negative fractions, 123, 124
Negative integers, 53
Negative numbers, 53, 124
Negative of a signed number, **67,** 123
Not greater than, 48
Not less than, 48
Notation, decimal, 184, 196
  functional, 167
  scientific, 196
  standard, 196
*n*th powers, **25**
*n*th root, **230,** 254
  principal, **230,** 235
Number, complex, **244**
  complex imaginary, 245
  composite, 96, 98
  consecutive natural, 18
  directed, 53
  even, 6, 234, 250
  imaginary, **243,** 245, 254
  irrational, 195, 228, **229,** 234
  literal, 6, 15, 140
  natural, 4, 55
  negative, 53, 124
  odd, 6, 234, 250
  positive, 53, 55
  prime, **96,** 97
  pure imaginary, 243, 245, 254
  rational, **125,** 195, 234, 252
  real, 234, **235,** 244
  signed, 53, 123
  transcendental, 229
  units, 96
  variable, 140, 165
  whose square is 2, 227–229
Number-pair, 169, 170, 221

- Number scale, 57, 68, 71, 106, 123, 131, 169
- Number system complex, 244
  - rational, 125
  - real, 234, **235**
- Number-triple, 225
- Numerator, **90,** 93, 94
- Numerical value, **59**

- Odd numbers, 6, 234, 250
- Operations, inverse, 35
  - order of, 22, 38, 82, 137
  - rational, 128, 141
- Order, of addition, 9
  - of multiplication, 13
  - of operations, 22, 38, 82, 137
- Ordinates, 171
- Origin, 106

- Parentheses, 8, 23, 83
  - omitting, in addition, 10, 62
    - in addition and multiplication 22
    - in addition and subtraction, 69
    - in addition, subtraction, multiplication, and division, 82
    - in multiplication, 14
    - in subtraction and multiplication, 38
  - unnecessary, 24
- Perfect square, trinomial, 146
- Periodic, repeating or, decimals, 194, 195
- Pi (π), 229
- Polar coordinates, 171
- Polynomials, **142, 192**
  - addition and subtraction of, 175
  - decimal, **185**
  - degree of, 142, 192
  - division of, **182**
  - divisor of, **182**
  - factors of, 144, 161
  - first-degree, 173
  - fourth-degree, factoring of, 161
  - highest common factor of, 151
  - least common multiple of, 151
  - multiplication of, 177
  - prime, 145
- Polynomials, products of, 138
  - standard form of, 143, 176
  - subtraction of, 175
  - terms of, 142
- Positive fractions, 123, 124
- Positive integers, 53, 55
- Positive numbers, 53, 55
- Postulate of closure, for addition, **5,** 136
  - for multiplication, **12**
- Powers, *n*th, **25**
  - product of, 28, 29, 137, 189
  - quotient of, 132, 133, 137, 189, 190
- Prime, relatively, 101
- Prime factors, of composite numbers, 98, 144, 228
  - of polynomials, 144
- Prime number, **96,** 97
- Prime polynomials, 145
- Principal *n*th root, **230,** 235
- Product, 11, *see* Multiplication
  - equal to zero, 47
- Products of polynomials, type, 138
- Pure imaginary number, **243,** 245, 254
- Pythagorean theorem, 32, 228

- Quadratic, 142
- Quadratic binomial, 145
- Quadratic trinomial, 142, 146, **147**
- Quotient, **79, 180,** *see* Division

- Radicals, 230, 235
  - index of, 230, 239
  - multiplication and division of, 239
  - simplest form of, 238
- Radicand, 230
- Range of variable, 165, 168
- Rational algebraic expression, 141
- Rational approximations, 232, 254
- Rational exponents, **235**
- Rational number system, 125
- Rational numbers, **125,** 195, 234, 252
  - addition of, 128
  - division of, 127
  - greater than and less than for, 131
  - multiplication of, 126
  - subtraction of, 128

Rational operations, 128, 141
Rationalizing denominators, 241, 247
Real number system, 234, **235,** 244
Real part of complex number, 245
Reciprocals, **106,** 108
Rectangular coordinates, 171
Relatively prime, 101
Remainder, in division, of integers, **180**
  of polynomials, **182**
Repeating or periodic decimals, 194, 195
Right triangle, 32
Roots, cube, **230,** 254
  of equations, 202
  *n*th, **230,** 254
  square, **230,** 254
  table of, 233

Scientific notation, 196
Secondary denominators, 121
Secondary fractions, 121
Setting up equations, 16, 17
Sieve of Eratosthenes, 97
Sign, equals, 7
Signed fractions, 123
Signed integers, 53
Signed numbers, 53
  addition of, 56, **60,** 61, 62
  division of, 81, 126, 127
  multiplication of, **73, 76,** 126
  subtraction of, **63,** 65, 68
Signs, law of, for division, 81
  for multiplication, **73,** 74, **76,** 137
Simple fraction, 120
Simplest form, of complex number, 245
  of radicals 238
Simultaneous equations, 221, 222, 225
Solutions, of equations, 40, 63, 110, 112, **202**
  of simultaneous equations, 221, 222, 225
Square of *x*, 26
Square root, **230,** 254
  table of, 233
  of 2, 227–229
Squares, difference of two, 145, 160
Standard form of polynomial, 143, 176
Standard notation, 196
Subscripts, 140
Subtraction, of complex numbers, 246
  of fractions, 128
  of integers, 56, 57, 59, **63**
  of natural numbers, **36,** 37, 44, 52, 63
  of polynomials, 175
  of rational numbers, 128
  of signed numbers, **63,** 65, 68
  of zero, 45
Subtraction and multiplication, grouping in, 38
Subtrahend, 37, 176
Sum, *see* Addition
Sum and difference of cubes, 151
Symbols, reading, 8
  of grouping, 8
  for multiplication, 11

Table, of cube roots, 233
  of square roots, 233
Taking out common factor, 21, 30
Terms, degree of, 142
  like, 33
  missing, 175, 186
  of polynomial, 142
Transcendental number, 229
Trinomial, 142
  quadratic, 146, 147
Trinomial perfect square, 146

Unit, imaginary, **243**
Units number, 96
Unknowns, 15, 17, 223, 225

Values of functions, 167, 218
Variable, 140, 165
  dependent, **165**
  independent, **165**
  range of, 165, 168
Verbal problems, 15, 111, 223
Vinculum, 8

Word problems, 15, 111, 223

Zero, **44**, 53, 86, 107, 125, 234
  divided by a number not zero, 86
  division by, 87, 106
Zero, exponent, **190**
  greater than and less than for, 46
  multiplication by, 45
  product equal to, 47
  subtraction of, 45

J. & K. UNIVERSITY LIB.
Acc. No 42152
Date 16-2-1966

ALLAMA IQBAL LIBRARY
42152